Trabajo de Fin de Máster: Coisotropic reduction in Multisymplectic Geometry

Máster en Matemáticas Avanzadas

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UCM

Outline

Calculus of Variations

Take a fibered manifold

$$Y \xrightarrow{\pi} X,$$

with coordinates

$$(x^\mu, y^i) \xrightarrow{\pi} (x^\mu).$$

We want to find sections $\phi : X \to Y$ that extremize certain functional (the action)

$$S[\phi] := \int_X \mathcal{L}(x^\mu, \phi^i, \frac{\partial \phi^i}{\partial x^\mu}),$$

where $\mathcal{L}$ (the Lagrangian density) is an n-form on X, with $\dim X = n$.

For first order field theories, we can interpret

$$\text{Lagrangian dentisy} \sim \mathcal{L} : J^1\pi \to \bigwedge^n X;$$

$$\text{Action} \sim S[\phi] = \int_X \mathcal{L} \circ j^1\phi.$$

The first order variational problem

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Stationary sections will satisfy

$$\left.\frac{\mathrm{d}}{\mathrm{d}t}\right|_{t=0} S[\phi_t] = 0,$$

for every possible variation ϕ_t, $\phi_0 = \phi$. The Euler-Lagrange equations for ϕ are:

$$\text{Locally,} \quad \frac{\partial L}{\partial y^i} = \frac{\mathrm{d}}{\mathrm{d}x^\mu}\left(\frac{\partial L}{\partial z^i_\mu}\right),$$

$$\text{Intrinsically,} \quad (j^1\phi)^* \iota_\xi \Omega_{\mathcal{L}} = 0, \forall \xi \in \mathfrak{X}(J^1 Y),$$

where $\Omega_{\mathcal{L}}$ is the multisymplectic form of the theory, a closed $(n+1)$-form

$$\Omega_{\mathcal{L}} = d\left(\frac{\partial L}{\partial z^i_\mu}\right) \wedge dy^i \wedge d^{n-1}x_\mu - d\left(\frac{\partial L}{\partial z^i_\mu} z^i_\mu - L\right) \wedge d^n x.$$

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Symplectic Geometry $\sim$ Classical Mechanics

Multisymplectic Geometry $\sim$ Classical Field Theories

Symplectic Geometry

Definition (Symplectic manifold)

A symplectic manifold is a pair (M, ω), where M is an manifold, and $\omega \in \Omega^2(M)$ is a closed, non-degenerate, 2-form.

Definition

For a subspace $i : W \hookrightarrow T_x M$, define the symplectic orthogonal as

$$W^\perp := \{v \in T_q M,\ \omega(v, w) = 0, \forall w \in W\} = \ker i^* \circ \flat.$$

$$\text{Important submanifolds} \begin{cases} \text{Lagrangian, } T_x L = (T_x L)^\perp \\ \\ \text{Coisotropic, } (T_x N)^\perp \subseteq T_x N \end{cases}$$

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$$(M,\omega) \text{ symplectic} \to (TM,\tilde{\omega}) \text{ symplectic},$$

$$\tilde{\omega} = \flat_\omega^* \omega_M;\ \flat_\omega : TM \to T^*M \ \text{(contraction)}$$

Definition

- Hamiltonian vector field: $X_H \in \mathfrak{X}(M)$, $(H \in C^\infty(M))$ such that

$$\iota_{X_H}\omega = dH.$$

- Locally Hamiltonian vector field: $X \in \mathfrak{X}(M)$ such that

$$d\iota_X \omega = 0.$$

Theorem

A vector field $X : M \to TM$ is locally Hamiltonian if and only if it defines a Lagrangian submanifold of $(TM,\tilde{\omega})$.

Dynamics = Lagrangian submanifolds (Weinstein's creed)

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Given a coisotropic submanifold $i : N \hookrightarrow M$, the distribution

$$x \mapsto (T_x N)^{\perp}$$

is regular and involutive. Therefore, it arises from a maximal foliation $\mathcal{F}$.
Then,

Theorem

If $N/\mathcal{F}$ admits a smooth manifold structure such that $\pi : N \to N/\mathcal{F}$ defines a submersion ($N/\mathcal{F}$ is a quotient manifold), then there is an unique symplectic form ω_N on $N/\mathcal{F}$ such that

$$\pi^* \omega_N = i^* \omega.$$

Furthermore, if L is a Lagrangian submanifold in M that has clean intersection with N, $\pi(L \cap N)$ is a Lagrangian submanifold in $(N/\mathcal{F}, \omega_N)$

Coisotropic reduction

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Poisson brackets

Definition

(M, ω) symplectic manifold, $f, g \in C^\infty(M)$.

$$\text{Poisson bracket: } \{f, g\} = \omega(X_f, X_g).$$

- Jacobi indentity

$$\{f, \{g, h\}\} + \text{cycl.} = 0,$$

- Leibniz indentity

$$\{fg, h\} = f\{g, h\} + g\{f, h\}.$$

Theorem

A submanifold $N \hookrightarrow M$ is coisotropic if and only if

$$I_N = \{f \in C^\infty(M) : df = 0 \text{ on } N\}$$

defines a Poisson subalgebra of $(C^\infty, \{\cdot, \cdot\})$.

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Multisymplectic Manifolds

Definition

A multisymplectic manifold of order k is a pair (M, ω), where M is a smooth manifold, and ω is a closed $(k+1)-$form.

No non-degeneracy required

Definition

For $W \subseteq T_xM$, and $1 \leq j \leq k$ define the multisymplectic orthogonal as

$$W^{\perp,j} := \{v \in T_xM : \iota_{v \wedge w_1 \wedge \cdots w_j}\omega = 0, \ \forall w_1, \ldots, w_j \in W\}.$$

Important submanifolds $\begin{cases} j-\text{Lagrangian}, \ T_xL + \ker \flat_1 = (T_xL)^{\perp,j} \\ j-\text{Coisotropic}, \ (T_xN)^{\perp} \subseteq T_xN + \ker \flat_1 \end{cases}$

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Hamiltonian multivector fields and forms

$$(M,\omega) \text{ multisymplectic} \to \left(\bigvee_q M, \widetilde{\Omega}^q\right) \text{ multisymplectic}$$

$$\widetilde{\Omega}_q = \flat_q^* \Omega_M^{k+1-q}, \; \flat_q : \bigvee_q M \to \bigwedge^{k+1-q} M \text{ (contraction)}$$

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- Locally Hamiltonian multivector field: $U : M \to \bigvee_q M$ such that

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A multivector field $U : M \to \bigvee_q M$ is locally Hamiltonian if and only if it defines a $(k+1-q)$−Lagrangian submanifold in $\left(\bigvee_q M, \widetilde{\Omega}^q\right)$

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Coisotropic submanifolds

Given a k-coisotropic submanifold $i : N \hookrightarrow M$, we have

Proposition

The distribution $x \mapsto (T_xN)^{\perp,k} \cap T_xN \subseteq T_xN$ is involutive.

Thus, when it is regular, it arises from a foliation $\mathcal{F}$.

Theorem

When $N/\mathcal{F}$ admits a smooth manifold structure such that the projection $\pi : N \to N/\mathcal{F}$ defines a submersion ($N/\mathcal{F}$ is a quotient manifold), there exists an unique multisymplectic form ω_N on $N/\mathcal{F}$ such that

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What about projection of Lagrangian submanifolds?

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Let L be a manifold and $\mathcal{E}$ be a regular distribution on L. Define:

$$\bigwedge_r^k L = \{\alpha \in \bigwedge^k L : \ \iota_{e_1 \wedge \cdots \wedge e_r} \alpha = 0, \forall e_1, \dots, e_r \in \mathcal{E}\}.$$

$$\left(\bigwedge_r^k L, \Omega_L \right)$$ is a multisymplectic manifold

Definition

A multisymplectic manifold of type (k, r) $(M, \omega, W, \mathcal{E})$ is a multisymplectic manifold (M, ω) that is locally multisymplectomorphic to $\bigwedge_r^k L$.

$W \sim$ vertical distribution

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Let L be a smooth manifold, $i : Q \subseteq L$ be a submanifold, and $\mathcal{E}$ be a regular distribution. Then,

Proposition

$N := \bigwedge_r^k L\big|_Q$ defines a k-coisotropic submanifold.

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For $N = \bigwedge_r^k L\big|_Q$, where $TQ \cap \mathcal{E}$ has constant rank,

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An example of coisotropic reduction

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An important class of Lagrangian submanifold are given by closed forms, since horizontal k-Lagrangian submanifolds are locally the image of closed forms.

$$\begin{cases} N = \bigwedge_r^k L\big|_Q, \\ \alpha : L \to \bigwedge_r^k L. \end{cases} \xrightarrow{\text{Coisotropic reduction}} \begin{cases} N/\mathcal{F} = \bigwedge_r^k Q, \\ i^*\alpha : Q \to \bigwedge_r^k Q. \end{cases}$$

Theorem

In our example, k-Lagrangian submanifolds transversal to the vertical distribution reduce to k-Lagrangian submanifolds.

Projection of Lagrangian submanifolds (example)

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Definition

Let $(M, \omega, W, \mathcal{E})$ be a multisymplectic manifold of type (k, r). A submanifold $i : N \hookrightarrow M$ is called vertical if $W|_N \subseteq TN$.

Theorem

Let $(M, \omega, W, \mathcal{E})$ be a multisymplectic manifold of type (k, r), $i : N \hookrightarrow M$ be a vertical k-coisotropic submanifold, and $j : L \hookrightarrow M$ be a k-Lagrangian submanifold complementary to W. Then there is a neighborhood U of L in M, a submanifold $Q \hookrightarrow L$, a neighborhood V of L in $\bigwedge_r^k L$, and a multisymplectomorphism

$$\phi : U \to V$$

satisfying

a) ϕ *is the identity on* L;

b) $\phi(N \cap U) = \bigwedge_r^k L\big|_Q \cap V$.

Local characterization of vertical coisotropic submanifolds

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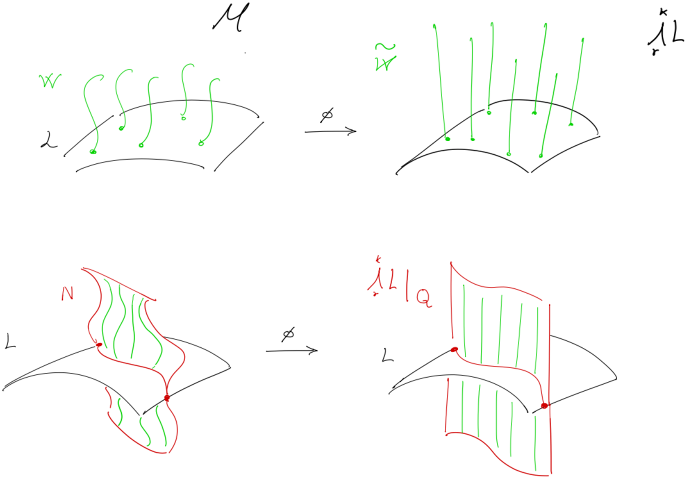
b) *$\phi(N \cap U) = \bigwedge_r^k L|_Q \cap V$.*

Idea of the proof



This local characterization allows us to prove:

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A general result is not possible, since we can easily find counterexamples.

Definition

Given two Hamiltonian forms $\alpha \in \Omega^{l_1}(M), \beta \in \Omega^{l_2}(M)$ on (M, ω),

$$\text{Poisson bracket: } \{\alpha, \beta\} := (-1)^{l_1 l_2 + 1} \iota_{X_\alpha \wedge X_\beta} \omega,$$

$$\iota_{X_\alpha} \omega = d\alpha, \ \iota_{X_\beta} \omega = d\beta.$$

- Well-defined (independent of the choice of X_α, X_β),
- Modulo closed-forms, it defines a graded Lie algebra on Hamiltonian forms

$$(-1)^{\deg \widehat{\alpha} \deg \hat{\gamma}} \{\hat{\alpha}, \{\hat{\beta}, \hat{\gamma}\}\} + \text{cycl.} = 0,$$

for

$$\hat{\alpha} := \alpha + (\text{closed forms}), \ \deg \hat{\alpha} := k - 1 - \operatorname{order}(\alpha).$$

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$$\iota_{X_\alpha} \omega = d\alpha,\ \iota_{X_\beta} \omega = d\beta.$$

- Well-defined (independent of the choice of X_α, X_β),
- Modulo closed-forms, it defines a graded Lie algebra on Hamiltonian forms

$$(-1)^{\deg \widehat{\alpha} \deg \hat{\gamma}} \{\hat{\alpha}, \{\hat{\beta}, \hat{\gamma}\}\} + \text{cycl.} = 0,$$

for

$$\hat{\alpha} := \alpha + (\text{closed forms}),\ \deg \hat{\alpha} := k - 1 - \operatorname{order}(\alpha).$$

- Restricts to a Lie bracket on

$$\widehat{\Omega}_H^{k-1}(M) := (\text{Hamiltonian } (k-1)-\text{forms})\big/(\text{closed } (k-1)-\text{forms})$$

Proposition

A k-coisotropic submanifold $i : N \hookrightarrow M$ defines a Lie subalgebra

$$I_N = \{\hat{\alpha} \in \widehat{\Omega}_H^{k-1}(M),\ i^* d\alpha = 0\}$$

of the Lie algebra $\widehat{\Omega}_H^{k-1}(M)$.

Final remarks and future research

Final remarks and future research

- We gave an interpretation of dynamics as Lagrangian submanifolds.
- We proved a coisotropic reduction theorem in a particular class of multisymplectic manifolds.
- For future research we have proposed the following:
  - Apply the results obtained to Field Theories (regularization, constraint analysis, etc)
  - Extend these results to multicontact geometry for the study of dissipative fields.
  - Connect these ideas to higher analogues of Dirac structures (giving a unified framework for both the Lagrangian and Hamiltonian formulation of Field Theory).

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Thank you for your attention!

Questions?