The graded Poisson bracket of general conservation laws in classical field theories

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Joint work with M. de León

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ICMAT

Structure of the talk

1. **Introduction to the problem**

2. **Graded Dirac structures**

3. **Dynamics on Graded Dirac manifolds**

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4. **Relation with the symplectic framework (work in progress)**

1. Introduction to the problem

The Poincaré–Cartan form in field theories

Take a configuration bundle over X (representing spacetime), together with its first jet bundle

$$J^1\pi \xrightarrow{\pi_{1,0}} Y \xrightarrow{\pi} X$$

(Locally think of $(x^\mu, y^i, y^i_{;\mu}) \mapsto (x^\mu, y^i) \mapsto (x^\mu)$).

A first order variational problem is now given by a Lagrangian density $\mathcal{L}\colon J^1\pi \to \bigwedge^n(\mathrm{T}^*X)$. The section solutions $\phi\colon X \to Y$ to the Euler–Lagrange equations are characterized geometrically by the Poincaré–Cartan form,

$$\Theta_{\mathcal{L}} = \left(L - y^i_{,\mu}\frac{\partial L}{\partial y^i_{,\mu}}\right)\mathrm{d}^n x + \frac{\partial L}{\partial y^i_{,\mu}}\mathrm{d}y^i \wedge \mathrm{d}^{n-1}x_\mu\,,$$

as those sections $\phi\colon X \to Y$ satisfying

$$-(j^1\phi)^*\iota_\xi \mathrm{d}\Theta_{\mathcal{L}} = 0\,, \quad \text{for all} \quad \xi \in \mathfrak{X}(J^1\pi)\,.$$

The algebra of conservation laws in field theories

Defining the pre-multisymplectic form $\Omega_{\mathcal{L}} := -\mathrm{d}\Theta_{\mathcal{L}}$, suppose that we have $\alpha \in \Omega^{n-1}(J^1\pi)$ such that

$$\mathrm{d}\alpha = \iota_{X_\alpha}\Omega_{\mathcal{L}}\,, \qquad \text{for some} \quad X_\alpha \in \mathfrak{X}(J^1\pi)\,. \tag{1}$$

Then we have that α defines a conservation law: $(j^1\phi)^*(\mathrm{d}\alpha) = 0$, for every solution ϕ of the Euler–Lagrange equations.

Given two $(n-1)$-forms $\alpha, \beta \in \Omega^{n-1}(J^1\pi)$ satisfying Eq. (1), we have that their Poisson bracket

$$\{\alpha, \beta\} := \iota_{X_\alpha \wedge X_\beta}\Omega_{\mathcal{L}}$$

satisfies again Eq. (1).

This defines a ~~Poisson~~ algebra of conservation laws.

The graded nature of the bracket

However, we may generalize Eq. (1) ($\mathrm{d}\alpha = \iota_{X_\alpha}\Omega_{\mathcal{L}}$) to

$$\alpha \in \Omega^a(J^1\pi) \qquad \text{and} \qquad X_\alpha \in \mathfrak{X}^{n-a}(J^1\pi)\,.$$

Arbitrary forms satisfying such equation will be called Hamiltonian forms. Let us denote by Ω^a_H the space of Hamiltonian a-forms.

Then, if α, β are Hamiltonian, so is their Graded Poisson bracket:

$$\{\alpha, \beta\} = (-1)^{n-1-b}\iota_{X_\alpha \wedge X_\beta}\Omega_{\mathcal{L}}\,.$$

So we propose the question:

Q: What is the role of this graded algebra in classical field theory?

A: It has to do with general conservation laws and observables

Previous work:

1. I. V. Kanatchikov. **"Canonical Structure of Classical Field Theory in the Polymomentum Phase Space".** In: *Rep. Math. Phys.* **41**.1 (1998), pp. 49–90
2. M. Á. Berbel and M. Castrillón-López. **"Poisson–Poincaré Reduction for Field Theories".** In: *J. Geom. Phys.* **191** (2023), p. 104879
3. F. Gay-Balmaz, J. C. Marrero, and N. Martínez-Alba. **"A New Canonical Affine Bracket Formulation of Hamiltonian Classical Field Theories of First Order".** In: *Rev. Real Acad. Cienc. Exactas Fis. Nat. Ser. A-Mat.* **118**.3 (2024), p. 103

2. Graded Dirac structures

Properties of the bracket

If we set $\deg_H \alpha := n - \deg \alpha$, then the Poisson bracket satisfies:

- *It is graded-skew-symmetric:* $\{\alpha, \beta\} = -(-1)^{\deg_H \alpha \deg_H \beta}\{\beta\alpha\}$.
- *It is local:* If $\mathrm{d}\alpha|_x = 0$, $\{\alpha, \beta\}|_x = 0$
- *It satisfies graded Jacobi identity (up to an exact term):*

$$(-1)^{\deg \alpha \deg \gamma}\{\{\alpha, \beta\}, \gamma\} + \text{cyclic terms} = \text{exact form}.$$

- *It satisfies Leibniz identity:* For $a = n - 1$, if $\beta \wedge d\gamma \in \Omega_H^{b+c-1}$, then

$$\{\beta \wedge \mathrm{d}\gamma, \alpha\} = \{\beta, \alpha\} \wedge \mathrm{d}\gamma + (-1)^{n-\deg \beta}\mathrm{d}\beta \wedge \{\gamma, \alpha\};$$

- *It is invariant by symmetries:* If $X \in \mathfrak{X}(M)$ and $\pounds_X \alpha = 0$, then $\iota_X \alpha \in \Omega_H^{a-2}$ and $\{\iota_X \alpha, \beta\} = (-1)^{\deg \beta}\iota_X\{\alpha, \beta\}$.

Graded Poisson brackets I

Let us study these brackets in general on a manifold M.

- Hamiltonian forms: α such that $\mathrm{d}\alpha \in S^{a+1}$, for some choice of subbundle $S^{a+1} \subseteq \bigwedge^{a+1}(\mathrm{T}^* M)$. Denote by Ω^a_H the space of such a-forms.
- These subbundles should be (surjectively) related by contractions:

$$S^n \xrightarrow{\iota_{\mathrm{T}M}} S^{n-1} \xrightarrow{\iota_{\mathrm{T}M}} \cdots \xrightarrow{\iota_{\mathrm{T}M}} S^1 \,.$$

(Think of $S^a := \iota_{\bigwedge^{n+1-a}(\mathrm{T}M)} \Omega_{\mathcal{L}}$).

Definition

A Graded Poisson bracket is a bilinear map $\Omega^a_H \otimes \Omega^b_H \xrightarrow{\{\cdot,\cdot\}} \Omega^{a+b-(n-1)}_H$ satisfying all the previous properties.

Graded Poisson brackets II

Is $\{\cdot,\cdot\}$ characterized by a tensorial object?

The case where $n = 1$ is true, such a bracket defines uniquely a Dirac structure.

Theorem (de León, I.L. 2025a)
Assume that S^n is locally generated by forms of constant coefficients. Let $K_1 \subseteq \mathrm{T}M, \ldots, K_n \subseteq \bigwedge^n(\mathrm{T}M)$ denote the annihilators of $S^1 \subseteq \mathrm{T}^*M, \ldots, S^n \subseteq \bigwedge^n(\mathrm{T}^*M)$, respectively. Then, there exists a unique family of maps

$$\sharp_a \colon S^a \to \bigwedge^{n+1-a}(\mathrm{T}M)/K_{n+1-a}$$

such that $\{\alpha,\beta\} = (-1)^{n-1-\deg\beta}\iota_{\sharp_{b+1}(\mathrm{d}\beta)}\mathrm{d}\alpha$.

Graded Dirac structures

Theorem (de León, I.L. 2025a (continued))
Furthermore, the maps $\sharp_a$ satisfy:

- *They are skew-symmetric:*
  $\iota_{\sharp_a(\alpha)}\beta = (-1)^{(n+1-a)(n+1-b)}\iota_{\sharp_b(\beta)}\alpha.$
- *They are integrable: The subbundles*

$$D^a = \left\{(\alpha, U) \in S^a \oplus_M \bigwedge^{n+1-a}(\mathrm{T}M) : \sharp_a(\alpha) = U + K_{n+1-a}\right\}$$

  are involutive under the graded Dorfmann bracket.

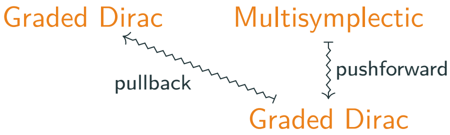
The converse also holds.

Definition (Graded Dirac structure*)
A graded Dirac structure on M is a family of maps $\sharp_a \colon S^a \to \bigwedge^{n+1-a}(\mathrm{T}M)/K_{n+1-a}$ satisfying the properties above.

The category of graded Dirac manifolds allows for pullbacks and pushforwards to be defined. In particular, we have natural examples:

- If (M, ω) is pre-multisymplectic, M/G, when G is a Lie group acting by symmetries, inherits a graded Dirac structure.
- If $\pi\colon Y \to X$ is a configuration bundle:

$$\begin{array}{ccc} J^1\pi & \xrightarrow{\mathrm{Leg}_{\mathcal{L}}} & \bigwedge_2^n Y \\ & \searrow^{\mathrm{leg}_{\mathcal{L}}} & \downarrow \\ & & \bigwedge_2^n Y / \bigwedge_1^n Y \end{array}$$



In general, it is not the pre-multisymplectic structure (but it is related). It is better suited for the study of internal symmetries and observables.

3. Dynamics on Graded Dirac manifolds

Fibered graded Dirac manifolds

Let $(M, S^a, \sharp_a)$ be a graded Dirac manifold of degree n and suppose that it is fibered over X (representing spacetime), $\tau\colon M \to X$.

Let us assume compatibility of the graded Dirac structure with the fibration in the following sense:

- $\dim X = n$.
- All τ-basic forms are contained in all S^a and S^a is comprised of $(a-1)$-horizontal forms.
- The $\sharp_a$ maps take value in the vertical distribution of the fibration.

We want to write equations for a section $\psi\colon X \to M$ as

$$\psi^*(\mathrm{d}\alpha) = (\mathrm{d}\alpha + \{\alpha, \mathcal{H}\}) \circ \psi\,, \qquad \text{for every} \qquad \alpha \in \Omega^{n-1}_H$$

However, degree considerations imply $\deg \mathcal{H} = n$ and the bracket is not defined for such forms.

Extensions of brackets I

This leads us to study extensions of graded Poisson brackets:

Theorem (de León, I.L. 2025b)
There exists a unique extension of $\{\cdot,\cdot\}$

$$\Omega^{n-1}_H \otimes \Omega^a_H[1] \to \Omega^a_H[1]$$

for arbitrary $a \geq 0$ *that satisfies the properties of* $\{\cdot,\cdot\}$.

Now, the expression

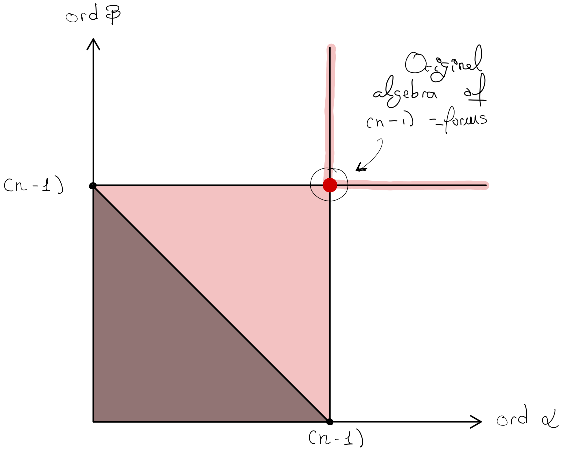
$$\psi^*(\mathrm{d}\alpha) = (\mathrm{d}\alpha + \{\alpha, \mathcal{H}\}) \circ \psi\,, \qquad \text{for every} \qquad \alpha \in \Omega^{n-1}_H$$

makes sense, for $\mathcal{H} \in \Omega^n_H[1]$ the Hamiltonian (a particular n-form that makes the right hand side semi-basic).

However, we would like for it to be defined for arbitrary Hamiltonian forms $\alpha \in \Omega^a_H$.

Extensions of brackets II

Current domain of definition:



How to extend it further?

Special Hamiltonian forms

Definition (Special Hamiltonian form)
A form $\alpha \in \Omega^a(M)$ is called special Hamiltonian if

$$\alpha \wedge \varepsilon \in \Omega_H^{n-1}\,,$$

for every closed and basic $(n-1-a)$-form ε.

If $\widetilde{\Omega}_H^a$ is the space of special Hamiltonian forms, we have $\widetilde{\Omega}_H^a \subseteq \Omega_H^a$ and it defines a subalgebra.

Theorem (de León, I.L. 2025b)
For $\alpha \in \widetilde{\Omega}_H^a$, and $\mathcal{H}$ a Hamiltonian, the expression $\{\alpha, \mathcal{H}\}$ is well defined and the following formula holds

$$\psi^*(\mathrm{d}\alpha) = (\mathrm{d}\alpha + \{\alpha, \mathcal{H}\}) \circ \psi\,,$$

for every $\psi\colon X \to M$ solving the equations defined by $\mathcal{H}$.

Technical remarks

The construction was based on a generalization of the $\sharp$ mapping associated to a graded Poisson bracket. In particular, we generalized the techniques employed in

1. P. W. Michor. **"A Generalization of Hamiltonian Mechanics".** In: *J. Geom. Phys.* **2**.2 (1985), pp. 67–82
2. J. Grabowski. **"Z-Graded Extensions of Poisson Brackets".** In: *Rev. Math. Phys.* **09**.01 (1997), pp. 1–27

to extend the brackets.

Properties of special Hamiltonian forms

Under integrability conditions on the PDE defined by $\mathcal{H}$ we have:

- An a-form α is special Hamiltonian if and only if it has well defined evolution: There exists a semi-basic $\beta \in \Omega^{a+1}(M)$ such that
$$\psi^*(\mathrm{d}\alpha) = \beta \circ \psi\,,$$
for every solution $\psi\colon X \to M$ of the equations.
- If α and β are semi-basic special Hamiltonian forms, $\alpha \wedge \beta$ is special Hamiltonian.
- If α is special Hamiltonian, there exists a multivector field $U_\alpha \in \mathfrak{X}^{n-a}(M)$ such that
$$\text{Important!} \to \sharp_n(\mathrm{d}\alpha \wedge \varepsilon) = \iota_\varepsilon U_\alpha + K_1\,,$$
for every closed and basic $(n-1-a)$-form ε.

Relation with higher form symmetries

- For α special Hamiltonian, there is U_α: $\sharp_n(\mathrm{d}\alpha \wedge \varepsilon) = \iota_\varepsilon U_\alpha + K_1$.
- If $\beta \in \Omega^{n-1}_H$ and $X \in \mathfrak{X}(M)$ are such that $\sharp_n(\mathrm{d}\alpha) = X + K_1$, we have that X defines a symmetry of the graded Dirac structure.

Theorem (de León, I.L. 2026)
If $\alpha \in \Omega^a$ has well defined evolution, there is a multivector field U_α such that $\iota_\varepsilon U_\alpha$ is a symmetry, for every closed and basic $(n-1-a)$-form ε. Or in other words, we have a symmetry parametrized by closed forms on X, namely a

$$(n-1-a)\text{-form symmetry}.$$

The graded Dirac structure on $J^1\pi$

Given a fibered graded Dirac manifold $\tau\colon M \to X$ and a Hamiltonian $\mathcal{H}$:

- There is a subalgebra of special Hamiltonian forms $\widetilde{\Omega}^a_H \subseteq \Omega^a_H$.
- This subalgebra is precisely comprised of forms with defined evolution.

Now, if $\mathcal{L}$ is a Lagrangian density, endowing $J^1\pi$ with the induced graded Dirac structure by $\mathrm{leg}_{\mathcal{L}}$,

- The Poincaré–Cartan form $\Theta_{\mathcal{L}}$ is a Hamiltonian.
- The equations $\psi^*(\mathrm{d}\alpha) = (\mathrm{d}\alpha + \{\alpha, \Theta_{\mathcal{L}}\}) \circ \psi$ are precisely the Euler–Lagrange equations.

Last remarks

- $\Omega_{\mathcal{L}}$ induces the algebra of Conservation laws.
- By studying the properties of this (graded) bracket we arrive naturally at graded Dirac geometry.
- When endowing $J^1\pi$ with this structure, rather than the induced by $\Omega_{\mathcal{L}}$, we obtain:
  - The Poincaré–Cartan form still plays an important role: It can be thought of as the Hamiltonian, defining dynamics.
  - The algebra of Hamiltonian forms extends the previous algebra: it contains all forms with defined evolution.
  - These forms are related to higher form symmetries in the following way:

$$\text{Defined evolution} \rightarrow \text{Higher form symmetries of the geometry}\,,$$

$$\text{closed on solutions} \rightarrow \text{Higher form symmetries of } \Theta_{\mathcal{L}}\,.$$

Future (and ongoing) work

- Noether Theorem?
- Relation with the infinite dimensional symplectic framework?
- Relation to reduction, reconstruction?
- Relation to integrability?

References

Other important references

1. C. L. Rogers. **"L_∞-Algebras from Multisymplectic Geometry".** In: *Letters in Mathematical Physics* 100.1 (Apr. 1, 2012), pp. 29–50
2. F. Cantrijn, A. Ibort, and M. León. **"Hamiltonian Structures on Multisymplectic Manifolds".** In: *Rend. Sem. Mat.* 54 (Jan. 1996)
3. J. Vankerschaver, H. Yoshimura, and M. Leok. **"The Hamilton-Pontryagin Principle and Multi-Dirac Structures for Classical Field Theories".** In: *Journal of Mathematical Physics* 53.7 (July 13, 2012), p. 072903
4. H. Bursztyn, N. Martinez Alba, and R. Rubio. **"On Higher Dirac Structures".** In: *Int. Math. Res.* **2019**.5 (2019). 10.1093/imrn/rnx163, pp. 1503–1542

References the talk was based on

1. M. de León and R.I. **"Graded Poisson and Graded Dirac Structures".** In: *J. Math. Phys.* **66**.2 (2025). 10.1063/5.0243128, p. 022901
2. M. de León and R.I. ***A description of classical field equations using extensions of graded Poisson brackets*.** 2025. arXiv: `2507.04743 [math-ph]`
3. M. de León and R.I. ***The relation between the observables in the space of solutions and the multisymplectic framework*.** 2026 (work in progress)

Thank you for your attention and...

Happy Birthday to Juan Carlos!

4. Relation with the symplectic framework (work in progress)

The symplectic framework

Let $\pi\colon Y \to X$ be a fibre bundle and $\mathcal{L}\colon J^1\pi \to \bigwedge^n(\mathrm{T}^*X)$ be a Lagrangian density.

- $\Gamma(\pi)$ is a Fréchet manifold.
- The space of solutions to the Euler–Lagrange equations $\boldsymbol{Sols} \subseteq \Gamma(\pi)$ can be endowed with a pre-symplectic form $\boldsymbol{\omega}$.

In fact, from a space-time splitting $X = \mathbb{R} \times \Sigma^{n-1}$, the pre-symplectic structure can be defined as

$$\boldsymbol{\omega}|_{\phi}(\xi_1, \xi_2) = \int_{\Sigma} (j^1\phi)^* \left(\iota_{\xi_1 \wedge \xi_2} \Omega_{\mathcal{L}}\right) .$$

From this, we obtain a Poisson bracket on the space of admissible functionals $\mathscr{C}^{\infty}_{\mathrm{ad}}(\boldsymbol{Sols}, \mathbb{R})$.

A different characterization of special Hamiltonian forms

Let $\alpha \in \Omega^a_H$ be a Hamiltonian form (with respect to the Graded Dirac structure). Let A be a compact oriented a-dimensional manifold. Then, we have a natural map

$$\Phi_\alpha \colon \mathscr{C}^\infty(A, X) \to \mathscr{C}^\infty(\boldsymbol{Sols}, \mathbb{R})$$

given by integration, for $i \colon A \to X$, and $\phi \in \boldsymbol{Sols}$:

$$\Phi_\alpha(i)[\phi] := \int_A (j^1\phi \circ i)^* \alpha \,.$$

Theorem (de León, I.L. 2026)
The map Φ_α takes values in the space of admissible functionals if and only if α is special Hamiltonian.

Relation among the brackets

What is the relation between the brackets?

Theorem (de León, I.L. 2026)
Let $A^{(a)}$ and $B^{(b)}$ be compact embedded submanifolds of X and α, β be special Hamiltonian a and b-forms, respectively. Suppose that

- *$A = A_1 \cap \cdots \cap A_{\text{codim}\, A}$, for certain submanifolds A_σ of spatial codimension 1. Similarly, $B = B_1 \cap \cdots \cap B_{\text{codim}\, B}$, for B_σ with spatial codimension 1.*
- *Suppose that every pair of intersections $A_{\sigma_1} \cap B_{\sigma_1}$ is a clean intersection and $A_\sigma \cap B_\sigma = A \cap B$.*

Then, the following formula holds:

$$\Phi_{\{\alpha,\beta\}}(A \cap B) = \sum_{\sigma_1,\sigma_2} \{\Phi_\alpha(A_{\sigma_1}), \Phi_\beta(B_{\sigma_2})\}$$

Future (and ongoing) work

- Noether Theorem?
- Relation to reduction, reconstruction?
- Relation to integrability?

Other important references

1. C. L. Rogers. **“L_∞-Algebras from Multisymplectic Geometry”.** In: *Letters in Mathematical Physics* 100.1 (Apr. 1, 2012), pp. 29–50
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