

On the theory of G-structures applied to multisymplectic manifolds

OAK2: Observables, Actions and Quantization

Joint work with M. de León and M. Lainz

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Structure of the talk

1. Motivation
2. G -structures
3. Tableaux and Spencer cohomology
4. Multisymplectic structures as G -structures
5. Flatness of multisymplectic forms

Namely,

- (i) **Motivation**: The importance of calculating the space of Hamiltonian forms on a multisymplectic manifold.
- (ii) **G-structures**: Some generalities on G -structures.
- (iii) **Tableaux and Spencer cohomology**: Cartan tableaux, how to compute their extensions, and their relation to the integrability problem of G -structures.
- (iv) **Multisymplectic structures as G -structures**: Main application of the theory of G -structures; some results on multisymplectic Lie algebras.
- (v) **Flatness of multisymplectic forms**: We give tools to prove a Darboux theorem in general. For a fixed order j , we construct a linear model of a multisymplectic form that exhibits obstructions to integrability of order j .

1. Motivation

Hamiltonian forms and their relevance (I)

Let (M, ω) be a **multisymplectic manifold** (meaning $\omega \in \Omega^{k+1}(M)$ is closed and nondegenerate). The space of **Hamiltonian forms** is

$$\Omega_H^a(M) = \{\alpha \in \Omega^a(M) : d\alpha = \iota_{X_\alpha} \omega\}.$$

There exists a '**Poisson**' bracket $\Omega_H^a \otimes \Omega_H^b \xrightarrow{\{\cdot, \cdot\}} \Omega_H^{a+b-(k-2)}$

F. Cantrijn, A. Ibort, and M. León. "Hamiltonian Structures on Multisymplectic Manifolds". *Rend. Sem. Mat.* Jan. 1996

C. L. Rogers. " L_∞ -Algebras from Multisymplectic Geometry". *Letters in Mathematical Physics*, Apr. 1, 2012

M. Callies, Y. Frégier, C. L. Rogers, and M. Zambon. "Homotopy Moment Maps". *Advances in Mathematics*, 2016

F. Gieres. "Covariant Canonical Formulations of Classical Field Theories". *SciPost Physics Lecture Notes*, 2023

Hamiltonian forms and their relevance (II)

The geometry of these brackets is that of **higher Poisson structures** (although we usually restrict to a smaller category, namely **graded Poisson structures**).

M. Zambon. “ L_∞ -algebras and higher analogues of Dirac structures and Courant algebroids”. *Journal of Symplectic Geometry*, 2012

H. Bursztyn, N. Martinez Alba, and R. Rubio. “On Higher Dirac Structures”. *Int. Math. Res.* 2019

The bijective correspondence between ‘Poisson brackets’ as above and graded Poisson structures is given in:

M. de León and R.I. “Graded Poisson and Graded Dirac Structures”. *J. Math. Phys.* 2025

Hamiltonian forms and their relevance (III)

More recently, we have explored the relationship between the graded nature of the aforementioned brackets and **classical field theories**. In particular:

How to write dynamics, and the study of general (lower) conservation laws:

M. de Leon and R.I. “A Description of Classical Field Equations Using Extensions of Graded Poisson Brackets”. *Int. J. Geom. Methods Mod. Phys.* 2026

Relationship to the covariant phase space formalism, where multivector fields are understood as symmetries:

M. de Leon, R.I., J. Gaset, and M. D. Maceda. “Multivector field symmetries, conserved charges and Noether’s theorem”. *In preparation*, 2026

Main problem (I)

In short, Hamiltonian forms are important. However, the computation of $\Omega_H^a(M)$ is very difficult in general (and varies significantly from case to case). In fact, since there is a **one-to-one correspondence**

$$\{\text{Hamiltonian } (k-1)\text{-forms}\} \longleftrightarrow \{\text{Local symmetries of } (M, \omega)\},$$

the problem of computing $\Omega_H^a(M)$ is tightly related to the problem of finding *Darboux-like theorems*. Because of this, and the fact that multisymplectic geometry englobes very different geometries, the theory of **G-structures** is particularly well-suited. The main motivation for the use of G-structures was the following problem:

How do we compute the algebra of Hamiltonian forms in the case of constant coefficients?

Main problem (II)

However, we have found that the applications go beyond this problem. In particular, we deal with the **Integrability problem** of multisymplectic structures:

Let (M, ω) be a multisymplectic manifold. Is there a coordinate chart around every point that makes ω of **constant coefficients**?

We have found that in most scenarios, the main technical problem of proving flatness of a G -structure can be avoided in multisymplectic structures by employing the **Moser trick**, which can be understood in a unified way through this theory.

2. G -structures

The bundle of frames

Let M be a smooth manifold of dimension n and V be a real n -dimensional vector space. The **bundle of frames** is defined as

$$\mathcal{F}(M) := \{\text{linear isomorphisms } \phi: V \rightarrow T_x M\}.$$

It is naturally a $\mathrm{GL}(V)$ -principal bundle over $\pi: \mathcal{F}(M) \rightarrow M$.

Definition

Let $G \subseteq \mathrm{GL}(V)$ be a closed Lie group. A **G -structure** is a reduction of the frame bundle to a bundle with structure group G :

$$B_G \hookrightarrow \mathcal{F}(M).$$

G -structures (I)

If a geometric structure on a manifold M is defined by some sort of object on each $T_x M$ (a tensor, subspace, orientation...) such that for every x, y we have a structure-preserving isomorphism $T_x M \cong T_y M$, it can be viewed as a G -structure:

- (i) An almost symplectic structure $\longleftrightarrow$ An $Sp(n)$ -structure.
- (ii) A Riemannian structure $\longleftrightarrow$ An $O(n)$ -structure.
- (iii) A regular distribution $\longleftrightarrow$ A $Block(j, n - j)$ -structure.
- (iv) A parallelism $\longleftrightarrow$ An $\{e\}$ -structure.

G-structures (II)

Definition

Let $B_G \hookrightarrow \mathcal{F}(M)$ be a G -structure on M . A local frame

$$\phi|_x: V \xrightarrow{\sim} T_x M$$

is said to be **adapted** if it corresponds to a local section

$$\phi: M \rightarrow B_G \subseteq \mathcal{F}(M).$$

- (i) For an **almost symplectic structure**: Frames $\{Q_i, P^i\}$ such that $\omega = Q^i \wedge P_i$.
- (ii) For a **Riemannian structure**: Orthonormal frames.
- (iii) For a **regular distribution**: Frames $\{X_i, Y_j\}$ such that $\{X_i\}$ spans the distribution.
- (iv) For a **parallelism**: The unique adapted frame is the parallelism itself.

G-structures (III)

Let $\psi: V \rightarrow M$ be a local diffeomorphism. Then, the pushforward induces a frame $\phi|_x: V \xrightarrow{\sim} T_x M$ by $\phi|_x := (\psi_*)|_{\psi^{-1}(x)}$.

Definition

A G -structure B_G is said to be **locally flat** if there is an adapted frame induced by a local diffeomorphism $\psi: V \rightarrow M$.

- (i) For an **almost symplectic structure**: Locally flat if and only if $d\omega = 0$ (Darboux's theorem).
- (ii) For a **Riemannian structure**: Locally flat if and only if the curvature vanishes.
- (iii) For a **regular distribution**: Locally flat if and only if it is involutive (Frobenius's theorem).
- (iv) For a **parallelism**: Locally flat if and only if the frame is a coordinate frame.

3. Tableaux and Spencer cohomology

Cartan Tableaux (I)

A **Cartan tableau** is a subspace $\mathcal{A} \subseteq V^* \otimes W$, where V and W are finite-dimensional vector spaces. Intuitively, these represent **homogeneous** systems of first-order linear PDEs (with constant coefficients). The system is a PDE on maps $f: V \rightarrow W$, and reads as

$$d_v f \in \mathcal{A}, \quad \forall v \in V.$$

Namely, if $\{v_i\}$ and $\{w_k\}$ denote bases of V and W , respectively, and

$$\mathcal{A} = \left\{ a_i^k v^i \otimes w_k \text{ such that } \pi_k^i a_i^k = 0 \right\}, \quad \pi_k^i \in \mathbb{R},$$

the PDE it defines is

$$\pi_k^i \frac{\partial f^k}{\partial v^i} = 0, \quad \text{for some} \quad f \in \mathcal{C}^\infty(V, W).$$

Cartan Tableaux (II)

To find solutions to this PDE, we will try to solve the PDE **formally** as a sum

$$f = \sum_{j=0}^{\infty} f^{(j)},$$

where each $f^{(j)} \in \mathcal{C}^{\infty}(V, W)$ is a **polynomial solution** of degree j . In terms of multilinear algebra, a polynomial map of degree j $f^{(j)}: V \rightarrow W$, can be interpreted as an element of the tensor product of the **symmetric product** of V^* with W

$$f^{(j)} \in S^j(V^*) \otimes W : \quad f^{(j)}(v) = a_{i_1, \dots, i_j}^k v^{i_1} \odot \dots \odot v^{i_j} \otimes w_k$$

The space of elements $f^{(j)} \in S^j(V^*) \otimes W$ that solve the above PDE is called **the extension** of degree j .

Cartan Tableaux (III)

Definition

The j -th extension of $\mathcal{A}$, denoted by $\mathcal{A}^{(j)}$, is the set of polynomial solutions of degree j . Formally,

$$\mathcal{A}^{(j)} = \{f \in S^j(V^*) \otimes W : \iota_{v_{j-1}} \cdots \iota_{v_1} f \in \mathcal{A}, \forall v_i\}.$$

A tableau is called of **finite type** if $\mathcal{A}^{(j)} = 0$ for some finite j . If it is not of finite type, it is called of **infinite type**.

Example

A Lie algebra $\mathfrak{g} \subseteq V^* \otimes V$ can be considered as a Cartan tableau. Its prolongations

$$\mathfrak{g}^{(j)} \subseteq S^j(V^*) \otimes V$$

can be interpreted as the **polynomial symmetries** of degree j of the flat G -structure $V \times G \hookrightarrow \mathcal{F}(V)$.

Spencer Cohomology (I)

Let us define the **complex**

$$C^{l,j}(\mathcal{A}) := \bigwedge^l V^* \otimes \mathcal{A}^{(j)} \subseteq \bigwedge^l V^* \otimes S^j(V^*) \otimes W,$$

together with the operator

$\partial: \bigwedge^l V^* \otimes S^j(V^*) \otimes W \rightarrow \bigwedge^{l+1} V^* \otimes S^{j-1}(V^*) \otimes W$ given by the linearization of the **de Rham differential**:

$$\partial(\alpha \otimes (v^1 \cdots v^j) \otimes w) := \sum_i v^i \wedge \alpha \otimes (v^1 \cdots \widehat{v}^i \cdots v^j) \otimes w.$$

It can easily be shown that $\partial C^{l,j}(\mathcal{A}) \subseteq C^{l+1,j-1}(\mathcal{A})$. The spaces

$$H^{l,j}(\mathcal{A}) := \frac{\ker(\partial: C^{l,j}(\mathcal{A}) \rightarrow C^{l+1,j-1}(\mathcal{A}))}{\operatorname{Im}(\partial: C^{l-1,j+1}(\mathcal{A}) \rightarrow C^{l,j}(\mathcal{A}))}$$

are called the **Spencer cohomology groups**.

Integrability of G -structures (I)

The Spencer cohomology $H^{2,j}(\mathfrak{g})$ of interpreting a Lie algebra as a tableaux gives obstructions to integrability of order $(j + 1)$ for any G -structure. Let us begin with some definitions:

Definition

A G -structure B_G on M is called $(k + 1)$ -flat at x if there is a coordinate chart

$$\psi: V \rightarrow M$$

with $\psi(0) = x$ and such that the induced frame $\phi: M \rightarrow \mathcal{F}(M)$ is tangent to B_G to order k at x .

Intuitively, these are coordinates making the structure flat to order k at a particular point x .

Integrability of G -structures (II)

V. Guillemin. “The Integrability Problem for G -Structures”.

Transactions of the American Mathematical Society, 1965

Let B_G be a G -structure on M . Then, G acts on every space $H^{2,j}(\mathfrak{g})$ so that there are induced vector bundles

$$B_G[H^{2,j}(\mathfrak{g})] \rightarrow M.$$

Theorem (Guillemin)

A G -structure on M which is globally k -flat has a natural section of $B_G[H^{2,k}(\mathfrak{g})]$, denoted by c_k and called the **structure tensor of order k** . Then, the G -structure is $(k+1)$ -flat at x if and only if $c_k(x) = 0$.

These tensors are constructed via the **solder form** (and higher order analogues).

Integrability of G -structures (III)

A G -structure is **formally flat** at x if it is k -flat, for every k . We say that it is formally flat, if it is formally flat at every point. Formal flatness of G -structures can be decided in a **finite** number of steps by checking whether the previous sections vanish. The existence of true flat coordinates is more involved. In the analytic case, it follows. However, we do not know if there exists a counterexample in the smooth case to the following question:

Is every formally flat G -structure **locally flat**?

Some positive cases follow from results in the literature:

- (i) True if G acts on V **irreducibly** (or if the action is completely reducible).
- (ii) True if G is of **finite type**, namely $\mathfrak{g}^{(j)} = \{0\}$ for some j .

4. Multisymplectic structures as G-structures

The multisymplectic groups

Let $\omega \in \Omega^{k+1}(M)$ be a differential k -form. When does (M, ω) induce a **G-structure**?

Proposition

*A pair (M, ω) induces a G -structure if and only if it is of **constant linear type**; namely, there are linear isomorphisms*

$$\phi_{x,y}: T_x M \xrightarrow{\sim} T_y M$$

satisfying $\phi_{x,y}^ \omega|_y = \omega|_x$.*

In particular, there exists a **linear model** (V, ϖ) with $\varpi \in \bigwedge^k V^*$, and G is the stabilizer group:

$$G = G_\varpi = \{g \in \mathrm{GL}(V) : g^* \varpi = \varpi\}.$$

We call these groups **the multisymplectic groups**.

The multisymplectic Lie algebras

For most of our considerations, the Lie algebra of G_ϖ , denoted by $\mathfrak{g}_\varpi$, will be the **fundamental object**. It is given by:

$$\mathfrak{g}_\varpi = \{T \in V^* \otimes V : \iota_T \varpi = 0\},$$

where the contraction by a $(1, 1)$ tensor is given by

$$(\iota_T \varpi)(v_1, \dots, v_k) := \sum_{j=1}^{k+1} \varpi(v_1, \dots, v_{j-1}, T(v_j), v_{j+1}, \dots, v_k).$$

How can we understand the **extensions** $\mathfrak{g}_\varpi^{(j)}$?

Let $\ker \varpi = \{v \in V : \iota_v \varpi = 0\}$. Then, we have

$S^j(V^*) \otimes \ker \varpi \subseteq \mathfrak{g}_\varpi^{(j)}$, and there is a natural isomorphism

$$\mathfrak{g}_\varpi^{(j)} / (S^j(V^*) \otimes \ker \varpi) \cong$$

$\{\text{Hamiltonian } (k-1)\text{-forms of homogeneous degree } (j+1)\}.$

Computing Hamiltonian $(k - 1)$ -forms (I)

Let (M, ω) be a **flat** multisymplectic manifold with linear type $\varpi \in \bigwedge^{k+1} V^*$. We compute the space of (local) Hamiltonian forms as follows:

- (i) **Calculate the Lie algebra** $\mathfrak{g}_{\varpi}$. (In most relevant scenarios, it will be an abelian extension, direct product, quotient, etc., of known Lie algebras, which simplifies the subsequent steps).
- (ii) **Compute its extensions** $\mathfrak{g}_{\varpi}^{(j)}$. The set of homogeneous polynomial Hamiltonian forms is then given by the previous isomorphism.
- (iii) Try to **reconstruct** the smooth forms from their Taylor expansions.

Computing Hamiltonian $(k - 1)$ -forms (II)

As an example, take the **canonical multisymplectic vector space** (the linear model for $\bigwedge^{k+1} T^*M$), namely $(V := L \oplus \bigwedge^k L^*, \varpi)$. Then,

Theorem (de León, Lainz, I-L)

For $k \geq 2$, $\mathfrak{g}_\varpi$ is an **abelian extension** of $\mathfrak{gl}(n)$, where $n = \dim L$.

The **extensions** $\mathfrak{g}_\varpi^{(j)}$ are canonically isomorphic to the direct sum

$$\mathfrak{gl}(n)^{(j)} \oplus \ker \left(\partial: S^j(L^*) \otimes \bigwedge^k L^* \rightarrow S^{j-1}(L^*) \otimes \bigwedge^k L^* \right).$$

Hence, we reconstruct the **space of Hamiltonian $(k - 2)$ -forms** and find that the most general $(k - 2)$ -form up to a closed term is

$$\alpha = (A_{\mu_2 \dots \mu_k}(x) + B^{\mu_1}(x) p_{\mu_1 \mu_2 \dots \mu_k}) dx^{\mu_2} \wedge \dots \wedge dx^{\mu_k}.$$

Computing Hamiltonian $(k - 1)$ -forms (III)

The same mechanism can be applied to the linear model of covariant field theories: Given an epimorphism of vector spaces $\pi: Y \rightarrow X$, we build the **multisymplectic linear model of type (m, n, r)** as

$$\left(Y \oplus \bigwedge_r^n Y^*, \varpi \right),$$

where $\bigwedge_r^n Y^*$ denotes the space of forms α satisfying $\iota_{e_1 \wedge \dots \wedge e_r} \alpha = 0$ whenever $e_i \in \ker \pi$.

Theorem (de León, Lainz, I-L)

For $n \geq 2$ and $1 < r \leq m := \dim \ker \pi$, $\mathfrak{g}_\varpi$ is isomorphic to an **abelian extension** of $\mathfrak{bloct}(n, m)$.

We can use this result to obtain all local expressions.

Summary of the talk so far

- (i) Multisymplectic manifolds with a constant linear type may be considered as manifolds endowed with G_ϖ -structures.
- (ii) The algebra of Hamiltonian $(k - 1)$ -forms can be computed by calculating the extensions of the multisymplectic Lie algebra $\mathfrak{g}_\varpi$.
- (iii) Generically, we expect this Lie algebra to be trivial. Hence, most multisymplectic forms have a 'trivial' algebra of Hamiltonian forms (containing only linear coefficients) and hence (arguably), most multisymplectic forms to be not suited for the study of fields.

The methods presented here apply only to manifolds admitting Darboux coordinates. The objective of the next part of the talk is to discuss the general existence of such adapted coordinates.

5. Flatness of multisymplectic forms

The natural representation (I)

Recall that if the natural action of $G \subseteq \mathrm{GL}(V)$ on V is completely irreducible, the vanishing of the structure tensors are a **necessary and sufficient condition** for the existence of flat coordinates. This can give a (hard to apply in general) strategy to proving this:

- (i) Let $W \subsetneq V$ be a G -invariant subspace. Then, we have induced representations

$$\rho_W: G \rightarrow \mathrm{GL}(W) \quad \text{and} \quad \rho_{V/W}: G \rightarrow \mathrm{GL}(V/W).$$

Denote by G_W and $G_{V/W}$ the image of these morphisms.

- (ii) Take an arbitrary formally flat G -structure. The previous decomposition induces an **involutive distribution**.
- (iii) The G -structure induces a G_W -structure on each leaf (**leaf geometry**) and a $G_{V/W}$ -structure on the space of leaves (**geometry transversal to leaves**).

The natural representation (II)

- (iv) If every formally flat G_W and $G_{V/W}$ structures are flat, we can take coordinates on M adapted to the geometry of the leaves and transversal to the leaves.
- (iiv) **However**, these are not coordinates adapted to the original G -structure, as there could be "compatibility conditions". We may try to modify these coordinates to be flat coordinates on the original manifold, but this is hard in full generality.

Nevertheless, for **multisymplectic structures**, the obstructions for solving this last step can be understood with the **Moser trick**.

The natural representation (III)

Definition

A subspace $W \subseteq V$ of a multisymplectic vector space (V, ϖ) , $\varpi \in \bigwedge^{k+1} V^*$ is called ***j-isotropic*** if $\iota_{w_1} \wedge \dots \wedge \iota_{w_{j+1}} \varpi = 0$, for all $w_i \in W$.

Then, we have the following alternative:

Theorem (de León, Lainz, I-L)

Let $\varpi \in \bigwedge^k V^$ be multisymplectic. If the natural representation is not irreducible and does not decompose as a direct sum, then (at least) one of the following holds:*

- (i) *There is a proper ***invariant j-isotropic*** subspace.*
- (ii) *There is an invariant subspace W for which $W^{\perp, k} = \{0\}$ and the induced representation on the quotient V/W is of ***finite type***.*

Flatness results (I)

In view of the previous discussion, the alternative given above allows for a systematic study of existence of **flat coordinates**:

- (i) If the representation is irreducible, every formally flat G_{ϖ} -structure is locally flat.
- (ii) If the representation decomposes as a direct sum, it suffices to study each induced representation. In particular, a G_{ϖ} -structure is locally flat if and only if the induced $G_{\varpi}|_{W_1}$ and $G_{\varpi}|_{W_2}$ -structures are locally flat.
- (iii) The last two scenarios are difficult, in general. However, we may use the **Moser trick** in some scenarios to understand the precise obstruction to integrability.

Flatness results (II)

The Moser trick implies the following:

Proposition (Moser trick)

Let $\varpi \in \bigwedge^{k+1} V^*$ and let M be endowed with a G_{ϖ} -structure. Since G_{ϖ} acts on

$$S = \{\iota_v \varpi : v \in V\},$$

we have an induced G_S -structure on M (G_S being the group of linear maps preserving the subspace S). Then, a G_{ϖ} -structure which is formally flat is locally flat if and only if every formally flat G_S -structure is locally flat.

Namely, ω admits a chart in which it has constant coefficients if and only if there are coordinates in which the bundle $\{\iota_v \omega : v \in TM\}$ is generated by forms of constant coefficients.

Idea of the proof

- (i) Let M be endowed with a G_{ϖ} -structure (equivalently, a form ω with **constant linear type**).
- (ii) Then, the subbundle $\tilde{S} = \{\iota_v \omega\} \subseteq \wedge^k T^*M$ induces a G_S -structure.
- (iii) By hypothesis, since the ω (eq. the G_{ϖ} -structure) is formally flat, so is $\tilde{S}$ (eq. the G_S -structure) and thus, $\tilde{S}$ is **flat**.
- (iv) This means that $\tilde{S}$ is generated by forms with **constant coefficients**.
- (v) Now we employ the usual **Moser trick** and obtain Darboux coordinates for ω .

Flatness results (III)

In the **1-isotropic** case, we have the following: Let $\varpi \in \wedge^{k+1} V^*$ be a multisymplectic form which admits an invariant 1-isotropic subspace and a complementary k -isotropic (non necessarily invariant) subspace.

Theorem (M. de León, M. Lainz, R.I)

Then, for any G_{ϖ} -structure, locally there is a collection of k -forms α_i and a $(k+1)$ -form α such that the G_{ϖ} -structure is integrable if and only if

- (i) *The induced ρ_W and $\rho_{V/W}$ structures are integrable.*
- (ii) *The linear PDE $\alpha = d(f^i \alpha_i)$, with $f^i \in C^\infty(M)$, has at least one solution.*

The PDE below can always be reduced to a PDE with **constant coefficients**.

Flatness results (IV)

Something similar follows for the *j-isotropic case*: Let ϖ be such that it has an invariant j -isotropic subspace admitting a (not necessarily invariant) $(j + 1 - k)$ -isotropic complement:

Theorem

For any G_{ϖ} -structure, there are local m -forms $\alpha_{i_1, \dots, i_m}$, for $m = 0, \dots, j$ such that the G_{ϖ} -structure is integrable if and only if:

- (i) *The induced ρ_W and $\rho_{V/W}$ -structures are integrable and:*
- (ii) *The PDE on functions $f^i \in C^\infty(M)$:*

$$\sum_{k=0}^{m-j} \binom{m+k}{m} df^{i_{m+1}} \wedge \dots \wedge df^{i_{m+k}} \wedge \alpha_{i_1, \dots, i_{m+k}} = 0$$

has at least one solution.

Flatness results (V)

What about the case where there is an invariant W with $W^{\perp,k} = \{0\}$?

Proposition

The map induced by restriction

$$\Phi: \mathfrak{g}_{\varpi} \longrightarrow \text{End}(W)$$

is a monomorphism.

In particular,

Theorem (M. de León, M. Lainz, R.I.)

A formally flat structure G_{ϖ} -structure is integrable if and only if the induced ρ_W -structure is integrable.

Examples (I)

Let us study some examples. The linear models are primarily borrowed from:

L. Ryvkin. “Darboux type theorems in multisymplectic geometry”.
Accepted for publication in the Springer INdAM volume 'Poisson Geometry and Mathematical Physics', 2025

Disclaimer: Our study only guarantees the existence of tensorial obstructions (namely, the obstruction tensors coming from $H^{2,j}(\mathfrak{g}_{\varpi})$) for the linear types $\varpi \in \bigwedge^k V^*$ to which the method applies. Obtaining the precise obstructions is crucial (requiring a deep understanding of the stabilizer group G_{ϖ}) and heavily depends on ϖ .

Examples (II)

Multicotangent bundles

If $V = L \oplus \bigwedge^{k-1} L^*$ and $\varpi \in \bigwedge^k V^*$ is defined by

$$\varpi((l_1, \alpha_1), \dots, (l_k, \alpha_k)) := \sum_{j=1}^k (-1)^{j-1} \alpha_j(l_1, \dots, \widehat{l_j}, \dots, l_k),$$

we find that G_ϖ is an abelian extension of $\mathrm{GL}(L)$. In fact, when $k \geq 3$, it leaves invariant the subspace $W = \{0\} \oplus \bigwedge^{k-1} L^*$, which is easily seen to be 1-isotropic. In particular, a G_ϖ -structure is flat if and only if the induced $\rho_W(G_\varpi)$ and $\rho_{V/W}(G_\varpi)$ -structures are flat and the PDE above admits a solution. Since the first is of finite type and the second satisfies $\rho_{V/W}(G_\varpi) = \mathrm{GL}(L)$, we conclude that every formally flat G_ϖ -structure is locally flat (the PDE may be solved employing the Poincaré Lemma).

Examples (III)

Product type

Let (V_1, ϖ_1) and (V_2, ϖ_2) be multisymplectic vector spaces with $k \geq 3$. Then, on $V = V_1 \oplus V_2$, take the linear model

$$\varpi = \varpi_1 - \varpi_2 \in \bigwedge^k V^*.$$

It is easy to show that $G_\varpi \cong G_{\varpi_1} \times G_{\varpi_2}$, and thus we deduce that a Darboux theorem exists for ϖ if and only if it exists for ϖ_1 and ϖ_2 . If the ϖ_i are volume forms, it follows that linear models of the form

$$\varpi = x^1 \wedge \cdots \wedge x^k + y^1 \wedge \cdots \wedge y^k,$$

for $k \geq 3$, satisfy a Darboux theorem (and the requirement is simply that the induced distributions are involutive).

Obstructions of arbitrary order (I)

Most known Darboux theorems (at least to the authors) are of **first order**. It is then a natural question to ask:

Question: Is there a bound j for which a Darboux theorem is necessarily of order $\leq j$?

With the theory of G -structures in mind, this is equivalent to asking:

Question: Is there a bound j for which $H^{2,m}(\mathfrak{g}_{\varpi}) = \{0\}$ for all $m \geq j$ and all ϖ ? We answer this question in the **negative** by proving that for every $j \geq 0$, there is a linear model ϖ_j satisfying

$$H^{2,j}(\mathfrak{g}_{\varpi_j}) \neq \{0\}.$$

The construction of this linear model is explicit.

Obstructions of arbitrary order (II)

Let V be a $2j + 5$ dimensional vector space and $\{x_0, \dots, x_{j+1}, y_0, y_1, p^0, \dots, p^j\}$ denote a basis. Define the 3-form

$$\varpi := \sum_{i=0}^j p_i \wedge \alpha^i = \sum_{i=0}^j p_i \wedge (y^0 \wedge x^i + y^1 \wedge x^{i+1}) .$$

Then, we have

Theorem (de León, Lainz, R-I)

The Lie algebra $\mathfrak{g}_\varpi \subseteq V^ \otimes V$ satisfies $H^{2,j}(\mathfrak{g}_\varpi) \neq \{0\}$.*

Notice that this is a Theorem of **3-forms**. A similar result for **k-forms** can simply be obtained on a $(2j + k + 2)$ -dimensional vector space simply by multiplying by a volume element.

Conclusions and outlook

- (i) There are several relevant questions in the multisymplectic framework for field theories which can benefit from an application of the well developed theory of G -structures.
- (ii) In this work we took two main directions:
 - (a) The computation of the space of (polynomial) Hamiltonian forms as the set of extensions of $\mathfrak{g}_\varpi$, tangentially showing that (probably) most forms have trivial Lie algebra.
 - (b) The existence of Darboux coordinates: We have showed that the study of the natural representation of $\mathfrak{g}_\varpi$ gives a strategy to prove flatness theorems.
- (iii) Several aspects remain to be studied:
 - (a) What about a pair (ϖ, dH) ? Can this give a multisymplectic 'flowbox' theorem?
 - (b) What about Hamiltonian forms of strictly lower degree?

This talk was based on:

M. de León, M. Lainz, and R.I. “The Spencer cohomology and integrability of multisymplectic structures”. *To appear (soon?) in arXiv*, 2026

Merci beaucoup pour votre attention!

Questions?

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