On the Spencer cohomology and integrability of multisymplectic structures

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Joint work with M. de León and M. Lainz

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Contributions of this work

Preprint this talk is based on: M. de León, R. Izquierdo-López, and M. Lainz. ***The Spencer cohomology and integrability of multisymplectic structures*.** 2026. arXiv: `2607.05267 [math.DG]`

In this article we:

(i) Provide a general approach towards proving a "Darboux Theorem" for differential forms of higher degree than 2.

(ii) Provide a sequence of linear types $\varpi_j \in \bigwedge^{k+1} V^*$ for which a Darboux Theorem is necessarily of order j. Hence, the order of the conditions on flatness for a form (M, ω) may be of arbitrary order.

I will only focus on the first.

G-structures (I)

Let M be a smooth manifold of dimension n and V be a real n-dimensional vector space. The bundle of frames is defined as

$$\mathcal{F}(M) := \{\text{linear isomorphisms } \phi\colon V \to \mathrm{T}_x M\}\,.$$

It is naturally a $\mathrm{GL}(V)$-principal bundle over $\pi\colon \mathcal{F}(M) \to M$.

Definition
Let $G \subseteq \mathrm{GL}(V)$ be a closed Lie group. A G-structure is a reduction of the frame bundle to a bundle with structure group G:

$$B_G \hookrightarrow \mathcal{F}(M)\,.$$

G-structures (II)

If a geometric structure on a manifold M is defined by some sort of object on each $\mathrm{T}_x M$ (a tensor, subspace, orientation...) such that for every x, y we have a structure-preserving isomorphism $\mathrm{T}_x M \cong \mathrm{T}_y M$, it can be viewed as a G-structure:

(i) An almost symplectic structure $\longleftrightarrow$ An $\mathrm{Sp}(n)$-structure.

(ii) A Riemannian structure $\longleftrightarrow$ An $\mathrm{O}(n)$-structure.

(iii) A regular distribution $\longleftrightarrow$ A $\mathrm{Block}(j, n-j)$-structure.

(iv) A parallelism $\longleftrightarrow$ An $\{e\}$-structure.

The (linear) Spencer cohomology (sketched)

From the Lie subgroup $G \subseteq \mathrm{GL}(V)$, we obtain a matrix Lie algebra $\mathfrak{g} \subseteq V^* \otimes V$ and we can build the Spencer cohomology

$$H^{\ell,j}(\mathfrak{g}), \qquad \ell, j \geq 0.$$

The group G acts naturally on $H^{\ell,j}(\mathfrak{g})$ so that every G-structure $\pi\colon B \to M$ has naturally and associated vector bundle $B[H^{\ell,j}(\mathfrak{g})] \to M$.

This is the bundle whose sections represent differential obstructions to flatness of the G-structure.

The integrability problem

Let $\pi\colon B \to M$ be a G-structure. The central question is:

Is B locally isomorphic to the trivial (flat) G-structure on V?

To each G-structure one attaches, for $k = 0, 1, 2, \dots,$ structure tensors

$$c_k \in \Gamma\big(B[H^{2,k}(\mathfrak{g})]\big)\,,$$

the successive obstructions to finding coordinates that make it flat to higher and higher order (Guillemin).

(i) If B is flat, in particular, we would have $c_0 = c_1 = \cdots = 0$.

(ii) On the other hand, we say that B is formally flat if $c_k = 0$ for every k, namely, there is a "formal coordinate chart" (given by a power series) making it flat at every point.

Is every formally flat G-structure flat?

Examples of the integrability problem

(i) Sp(n)-structure: The only nonzero structure tensor is $c_0 = \mathrm{d}\omega$. Its vanishing (which by the above discussion is equivalent to formal flatness), guarantees flatness by the Darboux Theorem.

(ii) O(n)-structure: Again, only c_0 appears, but now is the curvature of the Levi–Civita connection. In this case formal flatness also implies flatness.

(iii) Block($j, n - j$)-structure: The first structure tensor c_0 is the curvature of the associated distribution. Hence, Frobenius Theorem states that formal flatness implies flatness.

(iv) $\{e\}$-structure: Formal flatness always implies flatness for a parallelism.

Flatness theorems

As we saw above, all theorems that are of the form

Meta-Theorem: *Conditions on geometric structure* implies *There is a chart in which this structure has constant coefficients*

are particular instances of the following conjecture (not known to be true or false in full generality):

Conjecture: If some G-structure is formally flat, it is flat.

Hence, to find "Darboux Theorems" in the context of multisymplectic geometry we need to prove that "If a G_ϖ-structure is formally flat, it is flat", where G_ϖ is the stabilizer of certain linear type $\varpi \in \bigwedge^{k+1} V^*$. Our main contribution is a result on the natural representation of G_ϖ on V which simplifies this procedure, and reduces the problem to two cases.

Importance of the representation

Why is the study of the natural representation of G_ϖ so important? The following is true for general G-structures:

Because a lot of instances of "formal flatness implies flatness" are known to be true given general conditions on this representation. Some of the most useful are the following:

(i) If G acts irreducibly on V, then it is true that formal integrability implies integrability.

(ii) Also, if the action of G on V is completely reducible (meaning sum of irreps), formal integrability implies integrability.

In particular, if the action of G_ϖ satisfies one of the above, the "Darboux theorem" holds (formal integrability implies integrability). Our contribution is to study the least favorable case.

Multisymplectic structures as G-structures

Fix $\varpi \in \bigwedge^{k+1} V^*$ and define

$$G_\varpi := \{g \in \mathrm{GL}(V) \colon g^*\varpi = \varpi\}.$$

Definition
(M, ω), $\omega \in \Omega^{k+1}(M)$, has constant linear type ϖ if for every $x \in M$ there is a linear isomorphism $\phi \colon V \to \mathrm{T}_x M$ with $\phi^*\omega|_x = \varpi$.

Equivalently, (M, ω) defines (and is defined by) a G_ϖ-structure.

Main result

Definition
For $W \subseteq V$, $W^{\perp,j} := \{v \in V : \iota_{v \wedge w_1 \wedge \cdots \wedge w_j} \varpi = 0\,,\ \forall w_i \in W\}$. W is *j-isotropic* if $W \subseteq W^{\perp,j}$.

Theorem (de León, Lainz, I.L.)
Let $\varpi \in \bigwedge^{k+1} V^*$ be multisymplectic. If the action of G_ϖ on V is neither irreducible nor a direct sum, then:

(i) *Either there is an invariant j-isotropic subspace, or*
(ii) *Or there is an invariant W with $W^{\perp,k} = \{0\}$.*

This yields four (non-exclusive) types of multisymplectic forms: irreducible, product, j-isotropic, and semi-finite. Each type carries its own strategy towards a Darboux theorem.

Examples

(i) Volume: Since $\mathrm{Sl}(n)$ acts irreducibly, volume forms are of irreducible type, and hence admit Darboux Theorem.

(ii) Multicontangent type: k-form on $V = L \oplus \bigwedge^{k-1} L^*$:

$$\varpi((l_1, \alpha), \dots, (l_k, \alpha_k)) = \sum_{j=1}^{k} (-1)^j \alpha_j (l_1, \dots, \widehat{l_j}, \dots, l_k)\,,$$

is of 1-isotropic type. In the article we show that 1-isotropic types always admit a Darboux Theorem.

(iii) Product type: If ϖ_1 and ϖ_2 are multisymplectic on L_1 and L_2 ($k \geq 3$), then $\varpi := \varpi_1 + \varpi_2$ on $V = L_1 \oplus L_2$ is of product type. In particular it admits a Darboux Theorem if and only if ϖ_1 and ϖ_2 do.

Conclusions and further work

We have:

(i) We provided a general treatment ob the subject of finding "flat coordinates" of differential forms.

(ii) Namely, via the theory of G-structures, we were able to show that the least favorable cases for the study of integrability reduce to two scenarios.

We are currently:

(i) Investigating "Hamiltonian flowbox" of PDEs, via this particular theory.

References

[1] X. Gràcia, J. de Lucas, X. Rivas, and N. Román-Roy. **“On Darboux Theorems for Geometric Structures Induced by Closed Forms”.** In: *Rev. Real Acad. Cienc. Exactas Fis. Nat. Ser. A-Mat.* **118**.3 (2024). 10.1007/s13398-024-01632-w, p. 131.

[2] M. de León, R. Izquierdo-López, and M. Lainz. ***The Spencer cohomology and integrability of multisymplectic structures.*** 2026. arXiv: `2607.05267 [math.DG]`.

[3] G. Martin. **“A Darboux Theorem for Multi-Symplectic Manifolds”.** In: *Lett. Math. Phys.* **16**.2 (1988). 10.1007/BF00402020, pp. 133–138.

[4] L. Ryvkin. **“Darboux Type Theorems in Multisymplectic Geometry”.** In: *Accepted for publication in the Springer INdAM volume ‘Poisson Geometry and Mathematical Physics’* (2025). arXiv:2503.03672. arXiv: 2503.03672.

Thank you for your attention!

Pre-print:

Are there bounds for the order of a Darboux Theorem?

All flatness results so far are of first order: the obstruction lives in $H^{2,0}(\mathfrak{g}_\varpi)$ (a closedness-type condition on ω).

Is there j_0 such that $H^{2,j}(\mathfrak{g}_\varpi) = 0$ for every $j \geq j_0$ and every ϖ?

We answer: no. For every $j \geq 1$ we build a nondegenerate 3-form ϖ_j on a space of dimension $2j + 5$ with

$$H^{2,j}(\mathfrak{g}_{\varpi_j}) \neq \{0\}\,,$$

so that a Darboux theorem for ϖ_j genuinely requires an order-j condition.

Sequence of counterexamples

The nondegenerate

$$\varpi_j := \sum_{i=0}^{j} p_i \wedge \left(y^1 \wedge x^i + y^2 \wedge x^{i+1}\right) \in \bigwedge^3 V^*\,, \qquad \dim V = 2j+5\,,$$

gives $H^{2,j}(\mathfrak{g}_{\varpi_j}) \neq \{0\}$. Explicitly, for $f \in \mathscr{C}^\infty(\mathbb{R})$, the 3-form on $\mathbb{R}^{5+2j}$

$$\omega_j = \sum_{i=0}^{j} \mathrm{d}p_i \wedge \left(\mathrm{d}y^1 \wedge \mathrm{d}x^i + \mathrm{d}y^2 \wedge \mathrm{d}x^{i+1}\right) - f(y^2)\,\mathrm{d}y^1 \wedge \mathrm{d}y^2 \wedge \mathrm{d}x^{j+1}$$

has $c_0 = \cdots = c_{j-1} = 0$, and $c_j = 0$ *iff* $\partial^j f/\partial(y^2)^j \equiv 0$.

The order of a Darboux theorem for multisymplectic forms is unbounded.

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