The graded Jacobi bracket in dissipative field theories

Gamma Seminar

Joint work with M. de León and X. Rivas.

05 March, 2026

ICMAT

Plan of the talk

1. **Herglotz variational principle**

2. **Conformal symmetries and Jacobi brackets**

3. **Pre-multisymplectizations and Multicontact structures**

4. **Dynamics and the graded algebra of dissipated forms**

Namely

1. **Herglotz variational principle:** A (very much non-exhaustive) introduction to action-dependent field theories. Just to motivate the problem and geometric structures.
2. **Conformal symmetries and Jacobi brackets:** Here we introduce the graded conformal transformations of an arbitrary differential form Θ, and define the graded Jacobi bracket.
3. **Pre-multisymplectization and Multicontact structures:** Then, still in an abstract context, we treat the subject of (pre-)multisymplectizations, and study the correspondence of the brackets in the homogeneous setting.
4. **Dynamics and the graded algebra of dissipated forms:** Finally, if time allows, we return to the original problem and use the graded Jacobi bracket to induce a bracket of dissipated forms.

This talk is based on:

M. de León, R. Izquierdo-López, and X. Rivas. ***Brackets in multicontact geometry and multisymplectization.*** To appear in Mediterranean Journal of Mathematics. 2026

Although some motivations and examples are part of ongoing work (mainly the non-co-oriented point of view). Also, how to define dynamics will be presented in a more general context.

1. Herglotz variational principle

Main references for action dependent field theories (from the multicontact point of view)

M. de León, J. Gaset, M. C. Muñoz-Lecanda, X. Rivas, and N. Román-Roy. **"Multicontact formulation for non-conservative field theories".** In: *Journal of Physics A: Mathematical and Theoretical* 56.2 (2023), p. 025201. DOI: 10.1088/1751-8121/acb575

M. de León, J. Gaset, M. C. Muñoz-Lecanda, X. Rivas, and N. Román-Roy. **"Practical introduction to action-dependent field theories".** In: *Fortschr. Phys.* (2025). 10.1002/prop.70000

J. Gaset, M. Lainz, A. Mas, and X. Rivas. **"The Herglotz variational principle for dissipative field theories".** In: *Geom. Mech.* **1**.2 (2024). 10.1142/S2972458924500060, pp. 153–178

Action dependent Lagrangians I

Let $\pi\colon Y \to X$ be a fiber bundle. We want to formalize the notion of (first order) action dependent Lagrangian. If (x^μ, y^i) denote fibered coordinates, this is achieved as follows. Let $\mathcal{L} = L(x^\mu, y^i, y^i_{;\mu}, s^\mu)\mathrm{d}^n x$ be a Lagrangian density, where the variables s^μ will denote the action density. Then, we have, that for a section $y^i(x)$ we may solve the equation

$$\frac{\partial s^\mu}{\partial x^\mu} = L(x^\mu, y^i, y^i_{;\mu}, s^\mu(x))\,,$$

so that

$$\int_X \mathcal{L} = \int_X L(x^\mu, y^i(x), y^i_{;\mu}(x), s^\mu(x))\mathrm{d}^n x = \int_{\partial X} s^\mu(x)\mathrm{d}^{n-1}x_\mu\,.$$

Action dependent Lagrangians II

This induces the following variational principle, called the Herglotz variational principle

Of all fields $y^i(x)$, find those that make stationary

$$\mathcal{J}[y^i(x)] := \int_X L(x^\mu, y^i(x), y^i_{;\mu}(x), s^\mu(x))\mathrm{d}^n x\,,$$

where $\frac{\partial s^\mu}{\partial x^\mu} = L(x^\mu, y^i(x), y^i_{;\mu}(x), s^\mu(x))$.

Proposition
If $y^i(x)$ (and hence $s^\mu(x)$) satisfy the previous variational principle, we have that they satisfy the Herglotz–Euler–Lagrange equations

$$\frac{\mathrm{d}}{\mathrm{d}x^\mu}\left(\frac{\partial L}{\partial y^i_{;\mu}}\right) - \frac{\partial L}{\partial y^i} = \frac{\partial L}{\partial s^\mu}\frac{\partial L}{\partial y^i_{;\mu}}\,, \frac{\partial s^\mu}{\partial s^\mu} = L\,.$$

Intrinsic Herglotz variational principle I

Intrinsically, we may work with the bundle

$$J^1\pi \times \bigwedge^{n-1} \mathrm{T}^* X \to X\,.$$

An action dependent Lagrangian will just be a fibered map

$$\mathcal{L}\colon J^1\pi \times \bigwedge^{n-1} \mathrm{T}^* X \to \bigwedge^{n} \mathrm{T}^* X\,.$$

With coordinates $(x^\mu, y^i, y^i_{;\mu}, s^\mu)$ this reads as before $\mathcal{L} = L(x^\mu, y^i, y^i_{;\mu}, s^\mu)\mathrm{d}^n x$.

Intrinsic Herglotz variational principle II

Of all sections $\phi \in \Gamma(\pi)$, find those that make stationary

$$\mathcal{J}[\phi] := \int_X \mathcal{L} \circ (j^1\phi, s)\,,$$

where $\mathrm{d}s = \mathcal{L} \circ (j^1\phi, s)$.

As in the case of first order conservative fields theories, there exists an n-form, called the multicontact form

$$\Theta_{\mathcal{L}} = \mathrm{d}s^\mu \wedge \mathrm{d}^{n-1}x_\mu - \frac{\partial L}{\partial y^i_{;\mu}}\mathrm{d}y^i \wedge \mathrm{d}^{n-1}x_\mu + \left(\frac{\partial L}{\partial y^i_{;\mu}} y^i_{;\mu} - L\right)\mathrm{d}^n x\,,$$

which can be used to give an intrinsic description of the equations.

Intrinsic Herglotz variational principle III

Under certain conditions on $\Theta_{\mathcal{L}}$, there exists a 1-form $\sigma_{\mathcal{L}}$, called the dissipation 1-form, satisfying

$$\sigma_{\mathcal{L}} \wedge \iota_R \Theta_{\mathcal{L}} = \iota_R \mathrm{d}\Theta_{\mathcal{L}}\,,$$

for certain class of vectors $R \in \mathscr{R}$ called Reeb vector fields. When it exists, it is locally $\mathcal{L} = \frac{\partial L}{\partial s^\mu}\mathrm{d}x^\mu$, and measures the dissipative nature of the equations.

Furthermore, the equations are written as

$$(j^1\phi, s)^*\Theta_{\mathcal{L}} = 0$$

$$(j^1\phi, s)^*\iota_\xi(\mathrm{d} + \sigma_{\mathcal{L}}\wedge)\Theta_{\mathcal{L}} = 0\,, \forall \xi \in \mathfrak{X}\left(J^1\pi \times \bigwedge^{n-1}\mathrm{T}^*X\right).$$

A multicontact structure in the articles is then defined so that it recovers the previous case, locally.

Principal objectives

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(i) Introduce a general notion of multicontact structure, that allows for a general (more flexible) treatment.

(ii) In particular, introduce a more geometric way of defining the dissipation 1-form.

(iii) Define a *graded* version of the Jacobi brackets, and study its main properties.

(iv) Finally, relate it back to the case of field theories.

2. Conformal symmetries and Jacobi brackets

The Schouten–Nijenhuis bracket of multivectors

Let M be a smooth manifold. Denote by $\mathfrak{X}^p(M) = \Gamma\left(\bigwedge^p \mathrm{T}M\right)$ the space of multivectors

Theorem
There is a unique bracket

$$\mathfrak{X}^p(M) \otimes \mathfrak{X}^q(M) \to \mathfrak{X}^{p+q-1}(M)$$

extending the Lie bracket of vector fields that satisfies

(i) *It defines a graded Lie bracket (with the degree shift* $\widetilde{\deg} U = \deg U - 1$*).*

(ii) *It satisfies a graded Leibniz identity* $[U, V \wedge W] = [U, V] \wedge W + (-1)^{q(p-1)} V \wedge [U, W]$*, where* $U \in \mathfrak{X}^p(M)$*,* $V \in \mathfrak{X}^q(M)$ *and* $W \in \mathfrak{X}^r(M)$ *This is called the Schouten–Nijenhuis bracket.*

Multivectors as conformal symmetries I

Let M be a smooth manifold, and $\Theta \in \Omega^n(M)$ be an arbitrary differential form.

Definition
We say that a multivector field $U \in \mathfrak{X}^p(M)$, with $p \leq n$, defines an infinitesimal conformal transformation if there exists another multivector field (the conformal factor) $V \in \mathfrak{X}^{p-1}(M)$ satisfying

$$\pounds_U \Theta = \mathrm{d}\iota_U \Theta - (-1)^p \iota_U \mathrm{d}\Theta = \iota_V \Theta \,.$$

Denote the space of conformal infinitesimal transformations as $\mathfrak{X}_H(M)$.

When $p = 1$, so that X is a vector field, we recover the usual condition, $\pounds_X \Theta = g \cdot \Theta$, where $V = g \in \mathscr{C}^\infty(M)$ is simply a function due to degree considerations.

Multivectors as conformal symmetries II

When $\Theta \in \Omega^n(M)$, it makes sense not only to consider $\ker_1 \Theta$, but to consider its higher kernels,

$$\ker_p \Theta = \left\{ u \in \bigwedge^p \mathrm{T}M \colon \iota_u \Theta = 0 \right\} .$$

Define $\ker \Theta := \bigoplus_{p \geq 1} \ker_p \Theta$. Then

Proposition
A multivector field $U \in \mathfrak{X}^p(M)$ is an infinitesimal conformal transformation if and only if

$$[U, V] \in \ker \Theta \,, \qquad \forall V \in \Gamma(\ker \Theta) \,.$$

Equivalently, it defines a graded derivation

$$\Gamma(\ker \Theta) \xrightarrow{[\cdot, U]} \Gamma(\ker \Theta) \,.$$

Proof.
It follows by employing the multivectors Cartan calculus

W. M. Tulczyjew. **"The Graded Lie Algebra of Multivector Fields and the Generalized Lie Derivative of Forms".** In: *Bull. Acad. Polon. Sci., Ser. Math. Astr. et Phys.* **22**.9 (1974), pp. 937–942

Indeed, $[U, V] \in \Gamma(\ker_{p+q-1} \Theta)$ if and only if

$$0 = \iota_{[U,W]}\Theta = (-1)^{(p-1)q} \pounds_U \iota_W \Theta - \iota_W \pounds_U \Theta = -\iota_W \pounds_U \Theta \,.$$

Hence, $[U, W] \in \Gamma(\ker_{p+q-1} \Theta)$, for every $W \in \Gamma(\ker_q \Theta)$ if and only if $\iota_W \pounds_U \Theta = 0$, for every $W \in \Gamma(\ker_p \Theta)$. This last condition implies $\pounds_U \Theta = \iota_V \Theta$, for some $V \in \mathfrak{X}^{p-1}(M)$. □

The Gerstenhaber algebra of infinitesimal conformal transformations

Proposition

Let $\mathfrak{X}^p_H(M)$ denote the space of conformal infinitesimal transformations. Then:

(i) *It is closed under the Schouten–Nijenhuis bracket.*

(ii) *It is closed under the wedge product.*

In particular, it closes a Gerstenhaber subalgebra.

Proof.

Again, use Cartan calculus of multivectors. □

The algebra of conformal Hamiltonian forms I

In (co-oriented) contact geometry, the Jacobi bracket of two functions $f, g \in \mathscr{C}^\infty(M)$ is defined as

$$\{f, g\} := -\iota_{[X_f, X_g]}\eta\,,$$

where X_f and X_g are the unique conformal symmetries of η satisfying $\eta(X_f) = -f$ and $(\eta(X_g) = -g)$. When the structure is non co-oriented, instead of functions (sections of $M \times \mathbb{R} \to \mathbb{R}$), we have a similar correspondence with sections of a different line bundle $E \to M$ and symmetries of $\ker \eta$. We will use this principle to define a correspondence

$$\{\text{Conformal Hamiltonian forms}\} \cong \{\text{Conformal infinitesimal transformations}\}.$$

The algebra of conformal Hamiltonian forms II

Let $\Theta \in \Omega^n(M)$ be an arbitrary differential form.

Definition
A differential form $\alpha \in \Omega^a(M)$ is called conformal Hamiltonian (with respect to Θ) if $\alpha = -\iota_{X_\alpha}\Theta$, for some (not necessarily unique) infinitesimal conformal transformation $X_\alpha \in \mathfrak{X}^{n-a}(M)$. Denote the space of conformal Hamiltonian forms by $\Omega^a_H(M)$.

By the analogy with contact geometry, we are then led to define the graded Jacobi bracket as

$$\{\alpha, \beta\} := -\iota_{[X_\alpha, X_\beta]}\Theta \, .$$

Theorem
The graded Jacobi bracket is well defined.

Proof.
Again, multivector Cartan calculus: □

Well definedness of the graded Jacobi bracket

Continuation of proof.
Since $[X_\alpha, X_\beta]$ is again an infinitesimal conformal transformation, we only need to check that $\{\alpha, \beta\}$ only depends on the values of α and β:

$$\begin{aligned} -\iota_{[X_\alpha, X_\beta]}\Theta &= -(-1)^{(p-1)q}\pounds_{X_\alpha}\iota_{X_\beta}\Theta + \iota_{X_\beta}\pounds_{X_\alpha}\Theta \\ &= -(-1)^{(p-1)q}\pounds_{X_\alpha}\iota_{X_\beta}\Theta + \iota_{X_\beta}\iota_{V_\alpha}\Theta \\ &= (-1)^{(p-1)q}(\iota_{V_\alpha} - \pounds_{X_\alpha})\iota_{X_\beta}\Theta \\ &= (-1)^{(p-1)q}(\pounds_{X_\alpha} - \iota_{V_\alpha})\beta\,. \end{aligned}$$

An (skew)symmetric argument shows the same for α. □

The Gerstenhaber algebra of conformal Hamiltonian forms I

In fact we have just proved that the Schouten–Nijenhuis bracket induces a bracket:

$$\begin{array}{ccc} \mathfrak{X}_H(M) \otimes \mathfrak{X}_H(M) & \xrightarrow{-\iota_\bullet \Theta} & \Omega_H(M) \otimes \Omega_H(M) \\ \Big\downarrow [\cdot,\cdot] & & \Big\downarrow \{\cdot,\cdot\} \\ \mathfrak{X}_H(M) & \xrightarrow{\quad -\iota_\bullet \Theta \quad} & \Omega_H(M) \end{array} .$$

The same holds for the wedge product, which induces the cup product of conformal Hamiltonian forms:

$$\begin{array}{ccc} \mathfrak{X}_H(M) \otimes \mathfrak{X}_H(M) & \xrightarrow{-\iota_\bullet \Theta} & \Omega_H(M) \otimes \Omega_H(M) \\ \Big\downarrow \wedge & & \Big\downarrow \vee \\ \mathfrak{X}_H(M) & \xrightarrow{\quad -\iota_\bullet \Theta \quad} & \Omega_H(M) \end{array} .$$

The Gerstenhaber algebra of conformal Hamiltonian forms II

Then, $(\Omega_H(M), \{\cdot,\cdot\}, \vee)$ closes a Gerstenhaber algebra

(i) The Jacobi bracket $\{\cdot,\cdot\}$ is graded skew-symmetric and $\vee$ is graded symmetric with the degree shift

$$\deg_H \alpha := n - 1 - \deg \alpha\,.$$

(ii) It satisfies the Graded Jacobi identity. For $\alpha \in \Omega^a_H$, $\beta \in \Omega^b_H$ and $\gamma \in \Omega^c_H$, we get

$$(-1)^{(n-1-a)(n-1-c)}\{\alpha,\{\beta,\gamma\}\} + \text{cyclic terms} = 0\,.$$

(iii) We have a Leibniz identity:

$$\{\alpha, \beta \vee \gamma\} = \{\alpha,\beta\} \vee \gamma + (-1)^{(n-2-a)(n-1-b)}\beta \vee \{\alpha,\gamma\}\,.$$

(iv) Finally, it is local, so that it satisfies a weak Leibniz identity:

$$\operatorname{supp}\{\alpha,\beta\} \subseteq \operatorname{supp}\alpha \cap \operatorname{supp}\beta\,.$$

A non-cooriented view

(i) $\{$A differential n-form $\Theta\} \leftrightarrow$
$\{$Codimension 1 'multi'-distribution $\mathcal{K} \subseteq \bigwedge^n \mathrm{T}M\}$.

(ii) $\{$Multivectors U such that $\iota_U \Theta = 0\} \leftrightarrow$
$\{$Multivectors U such that $U \wedge \bigwedge^{n-p} \mathrm{T}M \subseteq \mathcal{K}\}$.

(iii) $\{$Infinitesimal conformal transformations$\} \leftrightarrow$
$\{$Derivations of the graded algebra$\}$.

(iv) $\{$Gerstenhaber algebra of conformal Hamiltonian forms$\} \leftrightarrow$
$\{$Gerstenhaber algebra of graded derivations$\}$.

Some examples

(i) Multicontact *à la* Vitagliano : Let $\mathcal{D}$ be a n-codimensional distribution on a manifold M. Then, we get a 1 codimensional 'multidistribution´

$$\mathcal{K} := \mathcal{D} \wedge \bigwedge^{n-1} \mathrm{T}M\,.$$

(ii) k-contact structures: If $\mathcal{D} = \ker_1 \eta_1 \cap \cdots \cap \ker_1 \eta_k$, we get a differential form

$$\Theta = \eta_1 \wedge \cdots \wedge \eta_k\,.$$

This is the 'co-oriented' version of the case above.

(iii) Action dependent fields: Let $\pi \colon Y \to X$ be a fiber bundle and consider $M := \bigwedge_2^n \mathrm{T}^* Y \times \bigwedge^{n-1} \mathrm{T}^* X$, where $n = \dim X$. Then, we get a natural n-form

$$\Theta = \mathrm{d}s^\mu \wedge \mathrm{d}^{n-1} x_\mu - p_i^\mu \mathrm{d}y^i \wedge \mathrm{d}^{n-1} x_\mu - p\mathrm{d}^n x\,.$$

Some references for the examples above

(i) L. Vitagliano. **"L_∞-algebras from multicontact geometry".** In: *Diff. Geom. Appl.* **39** (2015). 10.1016/j.difgeo.2015.01.006, pp. 147–165.

(ii) J. de Lucas, X. Rivas, and T. Sobczak. **"Foundations on k-contact geometry".** 2409.11001. 2024.

(iii) Previous references.

Conclusions (of this section)

(i) Given an arbitrary differential *n*-form Θ, we may study its conformal transformations in the graded sense.

(ii) Multivectors $U \in \mathfrak{X}^p(M)$ such that $\pounds_U \Theta = \iota_V \Theta$ are closed both for the Schouten-Nijenhuis bracket and the wedge product.

(iii) *Dualizing* (contracting by Θ), we get an induced structure on a subclass of differential form, those that are conformal Hamiltonian.

(iv) This structure is a generalization of the usual Jacobi bracket to a graded bracket, and it recovers a Gerstenhaber algebra.

3. Pre-multisymplectizations and Multicontact structures

Definition
A manifold M together with a closed $(n+1)$-form Ω is called a pre-multisymplectic manifold. It is called multisymplectic if the map

$$\mathrm{T}M \to \bigwedge^{n} \mathrm{T}^* M\,, \qquad v \mapsto \iota_v \Omega$$

is a monomorphism.

Definition
A differential form $\widetilde{\alpha} \in \Omega^a(M)$ is called Hamiltonian if there exists a multivector $X_{\widetilde{\alpha}}$ satisfying $\iota_{X_{\widetilde{\alpha}}} \Omega = \mathrm{d}\widetilde{\alpha}$.

Definition
There is a bracket, called the graded Poisson bracket, of Hamiltonian forms: $\{\widetilde{\alpha}, \widetilde{\beta}\} = (-1)^{n-1-b} \iota_{X_{\widetilde{\alpha}} \wedge X_{\widetilde{\beta}}} \Omega$.

Pre-multisymplectic manifolds II

A particular (but non-defining) property of these brackets is that:

$$(-1)^{(n-1-a)(n-1-c)}\{\alpha,\{\beta,\gamma\}\} + \text{cyclic terms} = \text{exact term}.$$

In particular, when restricting the brackets to $(n-1)$-forms this defines what is known as an L_∞-algebra. However, these brackets satisfy much more interesting properties when one takes the graded nature of $\{\cdot,\cdot\}$ into account.

M. de León and R. Izquierdo-López. ***A description of classical field equations using extensions of graded Poisson brackets***. 2025. arXiv: 2507.04743 [math-ph]

Motivation I

For contact manifolds (M, η) we have a general procedure, called *symplectization*, which assigns to it a *homogeneous* symplectic manifold

$$\widetilde{M} := M \times \mathbb{R}_{\times}\,, \qquad \omega = -\mathrm{d}\upsilon\,, \qquad \upsilon = z \cdot \eta\,.$$

(More generally, it can be done in the non-cooriented context by defining the line bundle corresponding to the annihilator of the distribution).This correspondence defines an equivalence of:

$$\{\text{Contact manifold}\} \cong \{\text{Homogenous symplectic manifolds}\}\,.$$

This correspondence may be utilized to solve problems in either category by solving it in the other.

Motivation II

This may be understood more generally as:

$$\begin{array}{ccc} \{\text{Codimension 1 distributions}\} & \longleftrightarrow & \{\text{Homogeneous pre-symp.}\} \\ \uparrow & & \uparrow \\ \{\text{Contact distributions}\} & \longleftrightarrow & \{\text{Homogeneous symp.}\} \end{array}$$

We wish to have a similar picture for general 'multidistributions' (or differential forms Θ).

This is the notion of pre-multisymplectization. We will benefit from this procedure, as a lot of theory has been developed for pre-multisymplectic manifolds (and their brackets).

Pre-multisymplectizations

Let $\Theta \in \Omega^n(M)$ be an arbitrary differential form (or, more generally $\mathcal{K} \subseteq \bigwedge^n \mathrm{T}M$ a 1-codimensional subbundle).

Definition
The canonical (pre)-multisymplectization of (M, Θ) is the pre-multisymplectic manifold

$$\widetilde{M} := M \times \mathbb{R}_\times \,, \qquad \Omega = -\mathrm{d}\Upsilon \,, \qquad \Upsilon = z \cdot \Theta \,.$$

We can define more general pre-multisymplectizations, the theory works exactly the same. In fact, up to a universal cover of M, all pre-multisymplectizations'coincide´ (if the Liouville vector field is vertical). In the case of a 'multidistribution', we simply consider the line bundle of the annihilator, which embeds canonically in $\bigwedge^n \mathrm{T}^* M$, and has an induced pre-multisymplectic structure.

Lift of multivectors

Now we study the conformal–homogenous correspondence:

Theorem (de León, Rivas, I.L.)
Let $\Theta \in \Omega^n(M)$ be a differential form (res. $\mathcal{K}$ a multidistribution). Then, for every infinitesimal conformal transformation of Θ (res. $\mathcal{K}$), $U \in \mathfrak{X}^p(M)$ there exists a (up to $\ker_p \Omega$) unique multivector field $\widetilde{U} \in \mathfrak{X}^p(\widetilde{M})$ such that

(i) $\pounds_{\widetilde{U}} \Upsilon = 0$.

(ii) *It projects onto U using the canonical projection $\tau \colon \widetilde{M} \to M$.*

The multivector $\widetilde{U}$ is called the lift of U.

In the case of the canonical pre-multisymplectization it follows very easily, we have $\widetilde{U} = U + zV \wedge \frac{\partial}{\partial z}$. For general pre-multisymplectizations, some work needs to be done.

Similarly, we obtain the correspondence of conformal Hamiltonian forms in the base and Hamiltonian forms on the pre-multisymplectization.

Theorem (de León, Rivas, I.L)
Let $\alpha \in \Omega^a_H(M)$ be a conformal Hamiltonian form for Θ. Then, its homogenization $\widetilde{\alpha} = z \cdot \tau^* \alpha$ is a Hamiltonian form on $(\widetilde{M}, \Omega)$ and we recover the bracket as:

$$\{\widetilde{\alpha}, \widetilde{\beta}\} = \{\alpha, \beta\}^{\sim} + (-1)^{n-b} \mathrm{d}(\alpha \vee \beta)^{\sim} .$$

In order to obtain the co-oriented version, we can simply tensorize by the annihilator $\mathcal{K}^\circ$ of the bundle of multivectors.

Lift of conformal Hamiltonian forms II (L_∞-algebras parenthesis)

Notice that via the homogenization process, there is not a perfect correspondence of brackets. However, we may state the following equivalence:

Theorem (de León, Rivas, I.L.)
The maps

$$\Phi_m \colon \left(\Omega^{n-1}_H(M)\right)^{\otimes k} \to \Omega^{n-k}(M)\,,$$
$$\Phi_m(\alpha_1, \dots, \alpha_k) := (\alpha_1 \vee \cdots \vee \alpha_k)^{\sim}$$

define an L_∞-algebra isomorphism.

This, in fact, generalizes the correspondence defined by L. Vitagliano for n-dimensional distributions to general 'multidistributions' $\mathcal{K} \subseteq \bigwedge^n \mathrm{T}M$.

Multicontact forms I

Definition
We say that a differential form Θ (res. multidistribution) on M is multicontact if its canonical pre-multisymplectization is a multisymplectization.

Can we relate it back to a condition on Θ?

Proposition
The following two statements are equivalent to Θ being multicontact:

(i) Θ *is not in the image of the bundle map* $\flat_{\mathrm{d}\Theta} \colon \ker_1 \Theta \to \bigwedge^{n-1} \mathrm{T}^* M$ *where* $\flat_{\mathrm{d}\Theta}(v) = \iota_v \mathrm{d}\Theta$ *and* $\ker_1 \Theta \cap \ker_1 \mathrm{d}\Theta = \{0\}$.

(ii) *The following pairing is non-degenerate (so that* $\ker \Theta$ *is maximally nonintegrable):* $\ker_1 \Theta \otimes \ker \Theta \to \bigwedge \mathrm{T}M / \ker \Theta$, $v \otimes U \mapsto [v, U] + \ker \Theta$.

Multicontact forms II

Multicontact forms are invariant under conformal transformations, since it is ultimately a condition on the graded algebra $\ker\Theta$. We get the equivalence:

$$\begin{array}{ccc}
\{\text{Codim-1 dist. } \mathcal{K} \subseteq \textstyle\bigwedge^n \mathrm{T}M\} & \longleftrightarrow & \{\text{Homogeneous Pre-mult}\} \\
\uparrow\!\!\!\!\!\downarrow & & \uparrow\!\!\!\!\!\downarrow \\
\{\text{Multicontact}\} & \longleftrightarrow & \{\text{Homogeneous Mult}\}
\end{array}$$

Examples (Vitagliano's notion of multicontact and k-contact)

(i) A corank n distribution $\mathcal{D}$ on a manifold M is maximally nonintegrable if and only if the codim 1 multidistribution

$$\mathcal{K} := \mathcal{D} \wedge \bigwedge^{n-1} \mathrm{T}M$$

is multicontact.

(ii) When it is co-oriented, $\mathcal{D} = \ker_1 \eta_1 \cap \cdots \ker_1 \eta_n$, it is maximally nonintegrable if and only if the differential form

$$\Theta = \eta_1 \wedge \cdots \wedge \eta_k$$

is multicontact.

The example of action dependent fields

(i) The extended Hamiltonian formalism: If $\pi\colon Y \to X$ is a fiber bundle, the canonically n-form on the manifold $M = \bigwedge_2^n \mathrm{T}^* Y \times \bigwedge^{n-1} \mathrm{T}^* X$,

$$\Theta = \mathrm{d}s^\mu \wedge \mathrm{d}^{n-1}x_\mu - p_i^\mu \mathrm{d}y^i \wedge \mathrm{d}^{n-1}x_\mu - p\mathrm{d}^n x$$

is multicontact.

(ii) The reduced Hamiltonian formalism Also, on $M = (\bigwedge_2^n \mathrm{T}^* Y / \bigwedge_1^n \mathrm{T}^* Y) \times \bigwedge^{n-1} \mathrm{T}^* X$, (and for $n \geq 2$), the differential form

$$\Theta_h = \mathrm{d}s^\mu \wedge \mathrm{d}^{n-1}x_\mu - p_i^\mu \mathrm{d}y^i \wedge \mathrm{d}^{n-1}x_\mu + H\mathrm{d}^n x$$

is multicontact, for any $H \in \mathscr{C}^\infty(M)$.

Conclusions (again of this section)

(i) The pre-multisymplectization procedure defines a correspondence between conformal objects on (M, Θ) (or $(M, \mathcal{K})$) and homogeneous objects on $(\widetilde{M}, \Omega)$.

(ii) We may lift conformal multivectors to Hamiltonian multivectors.

(iii) Conformal Hamiltonian forms lift to Hamiltonian forms.

(iv) We have that the bracket are related (up to an exact term).

(v) This correspondence motivates the introduction of the notion of multicontact geometry as presented, which generalizes some notions present in the literature.

4. Dynamics and the graded algebra of dissipated forms

The Reeb multivector field I

Let (M, Θ) be a multicontact manifold. To define dynamics we will work with a fibered structure

$$\tau \colon M \to X \,,$$

which we assume compatible in the sense that Θ is (at most) 1-vertical. To define the Reeb multivector field, we will first need to define the Reeb distribution

$$\mathscr{R} := \{ v \in \mathrm{T}M : \iota_v \mathrm{d}\Theta = \text{semi-basic} \}.$$

We will assume $\mathscr{R}$ to be vertical.

Disclaimer: This is a bit more general than the work found in our article, but we have recently noticed that we may extend it as such.

The Reeb multivector field II

We then study the map

$$\flat_\Theta \colon \mathscr{R} \to \bigwedge^{n-1} \mathrm{T}^*X \oplus \bigwedge^{n} \mathrm{T}^*X\,, \qquad R \mapsto (\iota_R\Theta, \iota_R \mathrm{d}\Theta)\,.$$

This always defines an isomorphism onto its image $\mathcal{Z} \subseteq \bigwedge^{n-1} \mathrm{T}^*X \oplus \bigwedge^{n} \mathrm{T}^*X$. Then, we may interpret the inverse of this map as an element (modulo 2-vertical vector fields)

$$\flat_\Theta^{-1} \in \mathscr{R} \otimes \mathcal{Z}^* \cong \mathscr{R} \otimes \left(\left(\bigwedge^{n} \mathrm{T}X \oplus \bigwedge^{n-1} \mathrm{T}X \right) / \mathcal{Z}^\circ, \right)\,.$$

This is a contravariant object, and we define the Reeb 'multivector field' as $\mathcal{R} := (1, \frac{1}{n})\flat_\Theta^{-1}$.

Parenthesis: Relative multivectors and forms

There are natural pairings on the bundles

$$\bigwedge^{p+1} \mathrm{T}M \oplus \bigwedge^{p} \mathrm{T}M \qquad \text{and} \qquad \bigwedge^{a} \mathrm{T}^*M \oplus \bigwedge^{a+1} \mathrm{T}^*M\,.$$

This is the relative pairing:

$$\iota_{(U,V)}(\alpha, \beta) := (\iota_V \alpha + (-1)^b \iota_U \beta, \iota_V \beta).$$

We can also extend the notion of wedge product, Lie bracket, exterior differential..., and all have the same properties.
The Reeb multivector field is better understood as a relative n-multivector field.

The Reeb multivector field in the extended formalism

Let us work the example of the Reeb multivector field with

$$\Theta = \mathrm{d}s^\mu \wedge \mathrm{d}^{n-1}x_\mu - p_i^\mu \mathrm{d}y^i \wedge \mathrm{d}^{n-1}x_\mu - p\mathrm{d}^n x\,.$$

Then, the Reeb distribution is

$$\mathscr{R} = \ker_1 \mathrm{d}\Theta = \left\langle \frac{\partial}{\partial s^\mu}, \frac{\partial}{\partial p} \right\rangle\,.$$

And the isomorphism reads as $\frac{\partial}{\partial s^\mu} \mapsto (\mathrm{d}^{n-1}x_\mu, 0)$, $\frac{\partial}{\partial p} \mapsto (0, -\mathrm{d}^n x)$ so that we get that the (relative) Reeb multivector field is

$$\mathcal{R} = \frac{1}{n}\frac{\partial}{\partial s^\mu} \otimes \frac{\partial^{n-1}}{\partial^{n-1}x_\mu} - \frac{\partial}{\partial p} \otimes \frac{\partial^n}{\partial^n x} + \text{kernel terms}\,.$$

A quick computation shows that the kernel terms are trivial.

The Reeb multivector field in the reduced formalism

Now, if we take $H \in \mathscr{C}^\infty(M)$ a Hamiltonian and take the form

$$\Theta = \mathrm{d}s^\mu \wedge \mathrm{d}^{n-1}x_\mu - p_i^\mu \mathrm{d}y^i \wedge \mathrm{d}^{n-1}x_\mu + H\mathrm{d}^n x\,.$$

Then, the Reeb distribution is

$$\mathscr{R} = \ker_1 \mathrm{d}\Theta = \left\langle \frac{\partial}{\partial s^\mu} \right\rangle\,.$$

And the isomorphism reads as $\frac{\partial}{\partial s^\mu} \mapsto (\mathrm{d}^{n-1}x_\mu, \frac{\partial H}{\partial s^\mu}\mathrm{d}^n x)$ so that we get the Reeb multivector field

$$\mathcal{R} = \frac{1}{n}\frac{\partial}{\partial s^\mu} \otimes \frac{\partial^{n-1}}{\partial^{n-1}x_\mu} + \text{kernel terms}\,.$$

The kernel terms are non trivial now and are generated by

$$\frac{1}{n}\frac{\partial H}{\partial s^\mu}\frac{\partial^{n-1}}{\partial^{n-1}x_\mu} - \frac{\partial^n}{\partial^n x}\,.$$

The intrinsic dissipation form

Now, we may contract the relative multivector field $\mathcal{R}$ with the relative pair $(\Theta, \mathrm{d}\Theta)$. We get the following

Proposition
The contraction $\iota_{\mathcal{R}}(\Theta, \mathrm{d}\Theta) \in \mathbb{R} \oplus \mathrm{T}^* M$ *is well defined and*

$$\iota_{\mathcal{R}}(\Theta, \mathrm{d}\Theta) = \left(0, n(-1)^{n-1}\sigma_{\Theta}\right),$$

where σ_{Θ} *is a* 1*-form, which we refer to as the intrinsic dissipation* 1*-form.*

(i) In the extended case: $\sigma_{\Theta} = 0$.

(ii) In the reduced case: $\sigma_{\Theta} = \frac{\partial H}{\partial s^{\mu}} \mathrm{d}x^{\mu}$.

Hamiltonians and their dissipation form

Definition
A Hamiltonian is a semibasic form $h \in \Omega^n(M)$.

Definition
We have that there is a 1-form σ_h such that

$$\iota_{\mathcal{R}}(\mathrm{d}h, 0) = (n(-1)^{n-1}\sigma_h, 0)\,,$$

where σ_h is referred to as the dissipation form of the Hamiltonian.

For instance, in both the extended and reduced formalism,

$$\sigma_h = \frac{\partial \widetilde{H}}{\partial s^\mu}\mathrm{d}x^\mu\,,$$

for $h = \widetilde{H}\mathrm{d}^n x$.

Multicontact Hamilton–De Donder–Weyl equations

Let Θ be a multicontact form on a fibered manifold $M \to X$, in such a way that Θ is (at most) 1-vertical and the Reeb distribution $\mathscr{R}$ is vertical. Let $h \in \Omega^n(M)$ be a Hamiltonian. Then, define

$$\bar{\mathrm{d}}_h := \mathrm{d} + \sigma_\Theta \wedge + \sigma_h \wedge \ .$$

The Multicontact Hamilton–De Donder–Weyl equations for a section $\psi \colon X \to M$ are defined by

$$\psi^*(\Theta + h) = 0 \,, \qquad \psi^* \iota_\xi \bar{\mathrm{d}}_h(\Theta + h) = 0 \,, \forall \xi \in \mathfrak{X}(M).$$

Multisymplectization of the dynamics

Let $M \to X$ be fibered over X and Θ be a multicontact form which is (at most) 1-vertical. On its canonically multisymplecization $\widetilde{M} \to X$ we can define the homogenous Hamiltonian $\widetilde{h} = z \cdot h$. Assume the total dissipation form $\sigma_\Theta + \sigma_h$ to be exact.

Theorem (de León, Rivas, I.L.)
There is a $1-1$ correspondence between the solutions to the Multicontact Hamilton–De Donder–Weyl equations on (M, Θ) for the Hamiltonian h and the Hamilton–De Donder Weyl equations on $(\widetilde{M}, \Omega)$ for the Hamiltonian $\widetilde{h}$. Furthermore, if $\alpha \in \Omega_H^{n-1}(M)$ is conformal Hamiltonian we have that

$$\widetilde{\psi}^*(\mathrm{d}\widetilde{\alpha}) = \psi^*((\sigma_\Theta + \sigma_h) \wedge \alpha + \mathrm{d}\alpha)\,,$$

where $\widetilde{\psi}$ denotes the lift of the solution.

Dissipated forms

Definition
A form $\alpha \in \Omega^a(M)$ is said to be dissipated if

$$\psi^*(\mathrm{d}\alpha) = -(\sigma_\Theta + \sigma_h) \wedge \alpha\,,$$

for every solution of the equations ψ.

X. Rivas, N. Román-Roy, and B. M. Zawora. **"Symmetries and Noether's Theorem for Action-Dependent Multicontact Field Theories".** In: *Lett. Math. Phys.* **115**.5 (2025). 10.1007/s11005-025-01995-0, p. 108

Then, the previous result implies

Proposition
Dissipated forms correspond to closed (conserved) forms on solutions via the process of multisymplectization

The graded algebra of Dissipations forms

This, together with the result from

M. de León and R. Izquierdo-López. ***A description of classical field equations using extensions of graded Poisson brackets*.** 2025. arXiv: 2507.04743 [math-ph]

Theorem (de León, I.L.)
Closed forms define a graded subalgebra under the graded Poisson bracket

we obtain the immediate corollary

Theorem (de León, Rivas, I.L.)
Dissipated forms close a graded algebra under the graded Poisson bracket.

Conclusions

(i) We have given a more general treatment of multicontact structures, that can be seen both from the co-oriented and non co-oriented point of view.

(ii) In this general picture, we can introduce a graded bracket of forms, which generalizes the one present in contact geometry.

(iii) We can write the equations in a much more flexible setting, and also clarify the definitions of the dissipitation forms.

(iv) Together with the process of multisymplectization, we can prove that (graded) dissipation laws close a subalgebra under the graded Jacobi bracket.

Further work

As further work, we propose

(i) To prove a graded Noether theorem (work in progress in the multisymplectic setting).

(ii) To find the notion of 'graded Jacobi structures', which we believe will follow by employing the relative multivectors and forms.

(iii) In the more geometric side, define complements (coisotropic, Legendrian...).

References

[1] J. Gaset, M. Lainz, A. Mas, and X. Rivas. **"The Herglotz variational principle for dissipative field theories".** In: *Geom. Mech.* **1**.2 (2024). 10.1142/S2972458924500060, pp. 153–178.

[2] M. de León, J. Gaset, M. C. Muñoz-Lecanda, X. Rivas, and N. Román-Roy. **"Practical introduction to action-dependent field theories".** In: *Fortschr. Phys.* (2025). 10.1002/prop.70000.

[3] M. de León, J. Gaset, M. C. Muñoz-Lecanda, X. Rivas, and N. Román-Roy. **"Multicontact formulation for non-conservative field theories".** In: *Journal of Physics A: Mathematical and Theoretical* 56.2 (2023), p. 025201. DOI: 10.1088/1751-8121/acb575.

[4] M. de León and R. Izquierdo-López. ***A description of classical field equations using extensions of graded Poisson brackets.*** 2025. arXiv: 2507.04743 [math-ph].

[5] M. de León, R. Izquierdo-López, and X. Rivas. ***Brackets in multicontact geometry and multisymplectization.*** To appear in Mediterranean Journal of Mathematics. 2026.

[6] J. de Lucas, X. Rivas, and T. Sobczak. **"Foundations on *k*-contact geometry".** 2409.11001. 2024.

[7] X. Rivas, N. Román-Roy, and B. M. Zawora. **"Symmetries and Noether's Theorem for Action-Dependent Multicontact Field Theories".** In: *Lett. Math. Phys.* **115**.5 (2025). 10.1007/s11005-025-01995-0, p. 108.

[8] W. M. Tulczyjew. **"The Graded Lie Algebra of Multivector Fields and the Generalized Lie Derivative of Forms".** In: *Bull. Acad. Polon. Sci., Ser. Math. Astr. et Phys.* **22**.9 (1974), pp. 937–942.

[9] L. Vitagliano. **"L_∞-algebras from multicontact geometry".** In: *Diff. Geom. Appl.* **39** (2015). 10.1016/j.difgeo.2015.01.006, pp. 147–165.

Thank you for your attention!