HIGHER FORM SYMMETRIES AND GENERAL CONSERVATION LAWS

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Introduction and motivation

The geometric formulation of classical field theories uses **multisymplectic geometry**. Indeed, given a first-order Lagrangian density on a fiber bundle $\pi : E \to M$,

$$\mathcal{L} : J^1\pi \longrightarrow \bigwedge^n \mathrm{T}^*M,$$

one can build the **Poincaré–Cartan form**

$$\Theta_{\mathcal{L}} := \left(L - \frac{\partial L}{\partial u^i_{;\mu}} u^i_{;\mu} \right) \mathrm{d}^n x + \frac{\partial L}{\partial u^i_{;\mu}} \mathrm{d}u^i \wedge \mathrm{d}^{n-1}x_\mu \, .$$

The associated **multisymplectic form**, $\Omega_{\mathcal{L}} = -\mathrm{d}\Theta_{\mathcal{L}}$, encodes the full geometric information of the theory. In particular, it determines conserved quantities and their Poisson bracket. This bracket admits a natural graded generalization [1], but the role of this graded structure in the theory has not been fully explored. Clarifying this relation has been the main objective of our work [2].

The graded Poisson bracket

Let $(\mathcal{P}, \Omega)$, with $\Omega \in \Omega^{n+1}(\mathcal{P})$ closed, be a premultisymplectic manifold. A differential form $\alpha \in \Omega^a(\mathcal{P})$ is called **Hamiltonian** if $\mathrm{d}\alpha = \iota_{X_\alpha}\Omega$, for some multivector field $X_\alpha \in \mathfrak{X}^{n-a}(\mathcal{P})$, which we call Hamiltonian. Let $\Omega^a_H(\mathcal{P})$ denote the space of Hamiltonian forms. The **graded Poisson bracket** is defined as

$$\Omega^a_H(\mathcal{P}) \otimes \Omega^b_H(\mathcal{P}) \to \Omega^{a+b-(n-1)}_H(\mathcal{P}) \, ,$$

$$\{\alpha, \beta\} := (-1)^{n-1-b} \iota_{X_\alpha \wedge X_\beta} \Omega \, .$$

This bracket satisfies:

(i) It is *graded*, *graded-skew-symmetric* and satisfies a *graded Jacobi identity* (up to an exact term):

$$(-1)^{(n-1-a)(n-1-c)}\{\alpha, \{\beta, \gamma\}\} + \text{cyclic terms} = \text{exact term}$$

(ii) It satisfies a *graded Leibniz identity*: For $\alpha \in \Omega^{n-1}_H(\mathcal{P})$, we have

$$\{\beta^j \wedge \mathrm{d}\gamma_j \, , \alpha\} = \{\beta^j, \alpha\} \wedge \mathrm{d}\gamma_j + (-1)^{b_j - 1} \mathrm{d}\beta^j \wedge \{\gamma_j, \alpha\} \, .$$

(iii) It is *invariant* by symmetries: If $X \in \mathfrak{X}(\mathcal{P})$ and $\pounds_X \alpha = 0$, then

$$\{\iota_X \alpha, \beta\} = (-1)^{n-1-b} \iota_X \{\alpha, \beta\} \, .$$

Graded Dirac manifolds

The natural abstract context is that of **graded Dirac manifolds** [3], which are manifolds $\mathcal{P}$ together with a sequence of subbundles $S^1 \subseteq \mathrm{T}^*\mathcal{P}, \dots, S^n \subseteq \bigwedge^n \mathrm{T}^*\mathcal{P}$, and a bracket $\{\cdot, \cdot\}$ defined on the space of Hamiltonian forms

$$\Omega^{a-1}_H := \left\{ \alpha \in \Omega^{a-1}(\mathcal{P}) \colon \mathrm{d}\alpha \in S^a \right\}$$

satisfying all the above properties. It still makes sense to talk about the Hamiltonian multivector field associated to a Hamiltonian form, the one such that $\{\alpha, \beta\} = (-1)^{n-1-b} \iota_{X_\alpha} \mathrm{d}\beta$.

Forms with defined evolution and conservation laws

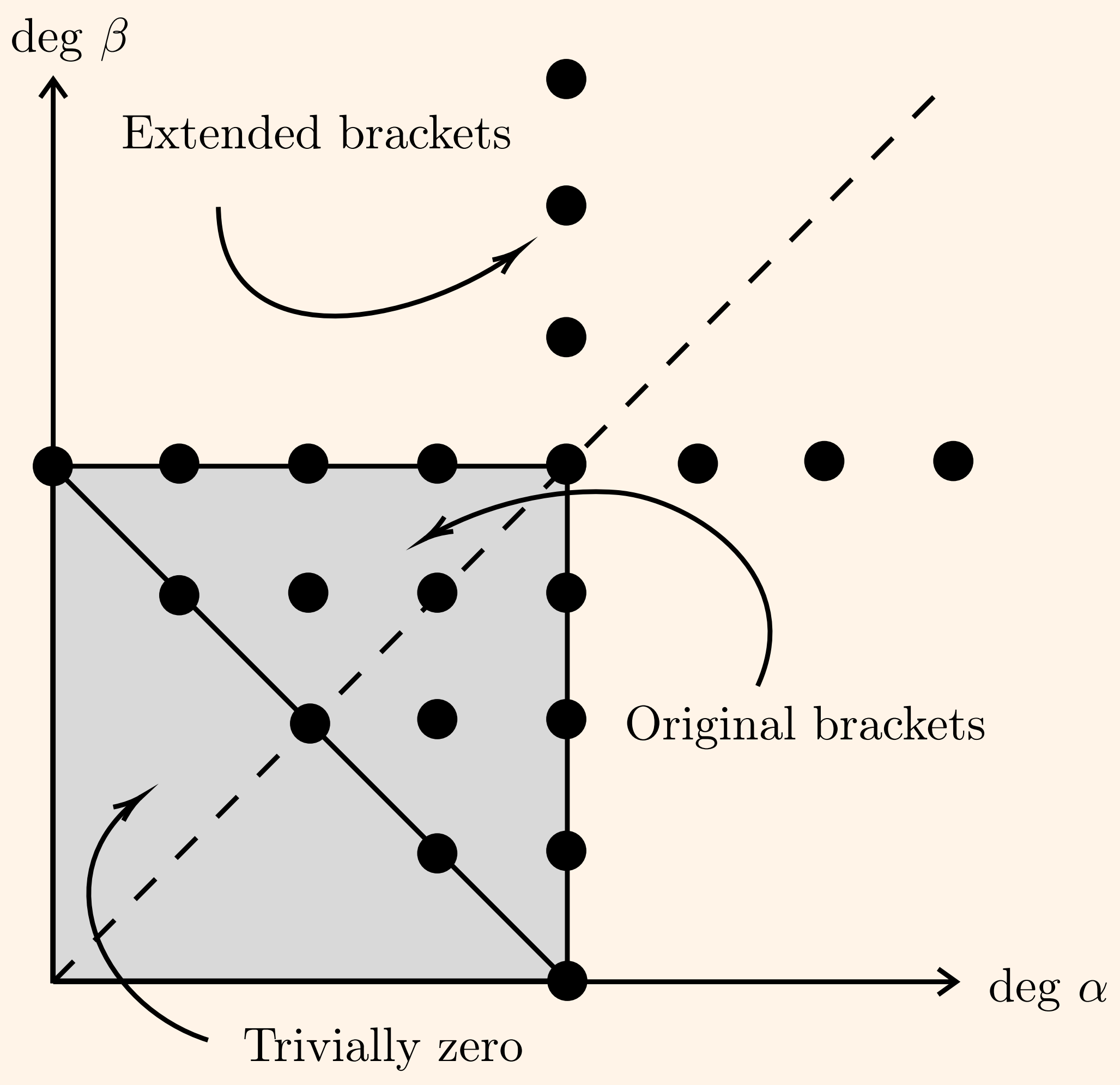
Let $\mathcal{P}$ be a graded Dirac manifold, fibered over a spacetime M, $\tau \colon \mathcal{P} \to M$. Let us assume that the graded Dirac structure is compatible with the fibration, meaning:

(i) The subbundles S^a consist of 1-vertical forms. In particular, Hamiltonian forms can be taken to be semibasic.

(ii) Basic forms $\varepsilon \in \Omega^a(M)$ are Hamiltonian and in the center of $\{\cdot, \cdot\}$, namely $\{\varepsilon, \alpha\} = 0$, for every α Hamiltonian.

Then, we can define dynamics in an abstract context employing the following result:

Theorem 1 (de León, I.L.). *Under the above conditions, there exists a family of forms* $\Omega^n_H(\mathcal{P})[1]$ *and an extension* $\Omega^{n-1}_H(\mathcal{P}) \otimes \Omega^{a-1}_H(\mathcal{P})[1] \to \Omega^{a-1}_H(\mathcal{P})[1]$ *extending the properties of* $\{\cdot, \cdot\}$, *where* $1 \leq a$ *is now unbounded.*



However, there is no canonical extension of the bracket in $\Omega^j_H(\mathcal{P}) \otimes \Omega^a_H(\mathcal{P})[1]$ for $j < n-1$. Nevertheless, it may be fixed by introducing *dynamics*. For a **Hamiltonian** $\mathcal{H} \in \Omega^n_H(\mathcal{P})[1]$, we can define dynamics for a section $\psi \colon M \to \mathcal{P}$ by the equations

$$\psi^*(\mathrm{d}\alpha) = (\mathrm{d}\alpha + \{\alpha, \mathcal{H}\}) \circ \psi \, , \tag{1}$$

for every $\alpha \in \Omega^{n-1}_H(\mathcal{P})$. There is a subclass of Hamiltonian forms *independent of* $\mathcal{H}$ that we call **special Hamiltonian** $\widetilde{\Omega}^a_H(\mathcal{P}) \subseteq \Omega^a_H(\mathcal{P})$ such that $\{\alpha, \mathcal{H}\}$ is *well-defined*, for every $\alpha \in \widetilde{\Omega}^a_H$.

Theorem 2 (de León, I.L.). *A form* $\alpha \in \Omega^a(\mathcal{P})$ *is special Hamiltonian if and only if* $\alpha \wedge \varepsilon \in \Omega^{n-1}_H(\mathcal{P})$, *for every closed and basic form* $\varepsilon \in \Omega^{n-1-a}(M)$.

In fact, for a special Hamiltonian form $\alpha \in \widetilde{\Omega}^a_H(\mathcal{P})$, we have that for every solution of the equations defined by Eq. (1) $\psi \colon M \to \mathcal{P}$, it holds that $\psi^*(\mathrm{d}\alpha) = (\mathrm{d}\alpha + \{\alpha, \mathcal{H}\}) \circ \psi$.

Theorem 3 (de León, I.L.). *For every special Hamiltonian form* $\alpha \in \widetilde{\Omega}^a_H(\mathcal{P})$, *there is a multivector field* $U_\alpha \in \mathfrak{X}^{n-a}(\mathcal{P})$ *(the associated* ***special Hamiltonian multivector field****) such that* $\iota_\varepsilon U_\alpha$ *is a Hamiltonian vector field for* $\alpha \wedge \varepsilon$, *for every closed and basic form* $\varepsilon \in \Omega^{n-1-a}(M)$.

The previous result implies that the associated multivector field may be thought of as a symmetry of the the Graded Dirac structure parametrized by closed forms, a **higher form symmetry**. The Hamiltonian form will be conserved (closed) if this symmetry is also a symmetry of $\mathcal{H}$.

Theorem 4 (de León, I.L.). *Special Hamiltonian forms close a subalgebra under the graded Poisson bracket. Furthermore, for a fixed Hamiltonian, closed forms on solutions also close a subalgebra.*

Electromagnetism

Let us study Electromagnetism on a Lorentzian manifold (M, g) with sources. The Lagrangian is:

$$\mathcal{L} = \left(-\frac{1}{4} F^{\mu\nu} F_{\mu\nu} + A_\mu J^\mu \right) \mathrm{d}^n x \, .$$

Here (A_μ, x^μ) denote fibered coordinates on $\mathrm{T}^*M \to M$, $F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu$ denotes the exterior derivative of $\mathbf{A} = A_\mu \mathrm{d}x^\mu$ and J^μ denotes the electric current. The graded Dirac structure in this case is generated by:

(i) All basic forms.

(ii) $A_\mu \mathrm{d}^{n-1}x_\nu - A_\nu \mathrm{d}^{n-1}x_\mu$.

(iii) $F^{\mu\nu}\mathrm{d}^{n-1}x_\mu$.

Proposition 1. *The algebra of special Hamiltonian forms are generated by basic forms, the vector potential* $\mathbf{A}$, *and the electric charge* $*\mathbf{F}$, *where* $\mathbf{F} = \mathrm{d}\mathbf{A}$.

For these generators, the graded Poisson bracket reads as $\{*\mathbf{F}, \mathbf{A}\} = n$. Their multivectors represent the electric and magnetic higher form symmetries, respectively.

Conclusions and References

In this work we found an algebra of special Hamiltonian forms associated to any Lagrangian, representing forms with well defined evolution. Furthermore, when restricted, it gives an algebra of (graded) conservation laws. The associated multivector fields may then be thought of as symmetries parametrized by closed forms. In the first case, symmetries of the geometry. In the second, symmetries of the Lagrangian.

References

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[2] M. de León and R. Izquierdo-López. *A description of classical field equations using extensions of graded Poisson brackets.* 2025. arXiv: `2507.04743 [math-ph]`.

[3] M. de León and R. Izquierdo-López. "Graded Poisson and graded Dirac structures". In: *J. Math. Phys.* **66**.2 (2025), p. 022901.