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A description of classical field equations using extensions of graded Poisson brackets

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Poisson brackets play a fundamental role both in mechanics and in classical field theories. In this paper, we develop a theory of extensions of graded Poisson brackets in graded Dirac manifolds. We then show how these extensions can be used to obtain the field equations of a particular theory as well as the evolution of forms of arbitrary degree, in a similar way that ordinary Poisson brackets provide in mechanics. To illustrate the theory, we apply the results to the study of regular Lagrangians and Yang–Mills theories.

Keywords: Graded Poisson brackets; classical field theories; graded Dirac structures; multisymplectic geometry.

1. Introduction

The finite-dimensional geometric setting for studying classical field theories is multisymplectic geometry, as has been established since the 1970s [25, 28, 37], just as symplectic geometry allowed the important developments of Hamiltonian dynamics. However, unlike in symplectic mechanics, in classical field theories we deal with differential forms of degree greater than 2, which is very difficult to study

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(for instance, forms of higher degree lack a Darboux theorem, see [33,44] for recent surveys on the matter). In particular, this allows us to introduce a *graded Poisson bracket*.

As far as we know, these brackets were first defined in [11], just as a continuation of the study of abstract multisymplectic manifolds carried in [12]. An important property of these brackets, already discussed in the work by Cantrijn, Ibort and de León, is that it induces a graded Lie algebra structure on $\Omega_H(M)/\mathrm{d}\Omega(M)$, the space of Hamiltonian forms modulo exact forms. The study of these brackets is far from being finished; indeed, a lot of recent research has been devoted to understand its interplay with field equations [2,14,27,35,36]. Up to now, this study is usually developed for particular cases, without identifying the essential properties of these brackets (as is usually done in Poisson geometry, for instance). Hence, it is an important issue to identity the defining properties of the brackets, which would allow for a generalization that should encode all constructions possible (as Poisson geometry does for Poisson brackets, for instance).

One possible approach for this generalization is that of L_∞-algebras. In [1], carrying a careful study of the exact terms present in the Jacobiator, Baez, Hoffnung and Rogers identifies that for the case of a multisymplectic manifold of order 2 (the case where the form has degree 3), the graded Poisson bracket induces what is known as a Lie-2-algebra. This result was later generalized to arbitrary multisymplectic manifolds in [42], proving that the graded Poisson bracket defines an L_∞-algebra. However, in order to obtain this structure, one must particularize the bracket at the highest degree of forms, only studying the Jacobiator when the forms have degree $n-1$, where $n+1$ is the degree of the multisymplectic form. A more complete study is developed in [20], where the defect of the Jacobi identity is studied for forms of arbitrary degree. However, L_∞ algebras lack a Leibniz identity, a property which is fundamental in classical mechanics and classical field theory.

In this vein, we recently study graded Poisson brackets on their own [18], defining a graded Poisson bracket as a graded-skew-symmetric bracket on a suitable subfamily of differential forms, called Hamiltonian. This bracket is required to satisfy a graded Jacobi identity up to an exact term, and two extra properties relating the bracket to exterior products and contractions (representing a graded version for the Leibniz identity, and invariance by symmetries, respectively). The first two properties recover the usual L_∞ algebra, and taking into consideration the last two has an important implication: the bracket is completely determined by a tensorial object. A manifold equipped with this object is called a *graded Dirac manifold* (or *graded Poisson manifold*, depending on the existence of geometric gauge as defined in [26,31]). In this paper, we continue the investigation of these structures and, in particular, a definition of dynamics in terms of these brackets is given.

Some preliminary results in this direction can be found, for instance, in the works by Kanatchikov [35,36], where the Poisson brackets are used to define dynamics and study evolution of observables. Nevertheless, the brackets defined depend heavily upon the choice of coordinates. A more intrinsic version, based upon the choice of a connection, can be found in the work by Berbel and Castrillón [2], where they employ the definition of dynamics to reduce the system. In all of these works, not only the brackets presented depend upon choices, but also *the Hamiltonian.*

More recently, in [27], Gay-Balmaz *et al.* developed a bracket description of the dynamics which is independent of these choices. In the multisymplectic description of field theories, fields are interpreted as sections of a bundle $Y \to X$, called the configuration bundle, where X is the base manifold, usually representing spacetime. In this setting, the Hamiltonian is interpreted as another section of a particular line bundle (see [13]):

$$h : \bigwedge_2^n Y / \bigwedge_1^n Y \to \bigwedge_2^n Y.$$

The bracket that the authors introduce is of the form $\{\alpha, h\}$, where α is a Hamiltonian $(n-1)$-form and it yields a semi-basic n-form over the base manifold X. This bracket is defined so that it computes the evolution of $(n-1)$-forms, in fact, a section $\psi : X \to \bigwedge_2^n Y / \bigwedge_1^n Y$ will be solution of the Hamilton–De Donder–Weyl equations determined by h if and only if it satisfies

$$\psi^*(\mathrm{d}\alpha) = \{\alpha, h\} \circ \psi, \quad \forall\, \alpha \in \Omega_H^{n-1}\left(\bigwedge_2^n Y / \bigwedge_1^n Y\right).$$

Linearity of this bracket is replaced by it being affine (as the space of Hamiltonians is affine), and the Jacobi identity is substituted by it defining an affine representation. The introduction of the previous operation raises some important questions:

(i) The fact that the Hamiltonians are defined as sections of a certain bundle presents some technical difficulties. The first one is how can one approach reduction by symmetries using such Hamiltonians. Would the reduced Hamiltonian be a different section of a different line bundle?

(ii) It is the case that the usual Poisson bracket of Hamiltonian $(n-1)$-forms recovers the classical Poisson bracket of the (infinite) dimensional instantaneous formalism via integration against a space-like slicing (see [3, 17]). How does this correspondence translate with the previous extension of the bracket?

(iii) In mechanics, the study of constraints, reduction, and quantization of a given system has benefited greatly from the introduction of abstract Poisson structures, as they make it possible to identify the key properties of the bracket that enable these procedures. It is therefore natural to ask: can one define a bracket that determines the dynamics of a classical field theory in a more general and flexible setting?

Although these are questions that one may in principle be able to answer in terms of the Hamiltonian section, we find that these are more naturally approached if the Hamiltonian is thought of as an n-form. Indeed, there is a one-to-one correspondence between sections of the previous line bundle and the family of n-forms $\Theta_h = h^*\Theta$, where Θ is the multisymplectic potential (also called the Liouville form) on $\bigwedge_2^n Y$.

Thus, if one identifies the Hamiltonian with the previous n-form $\mathcal{H} := \Theta_h$ the bracket reads as $\{\alpha, \mathcal{H}\}$, where now every object is a suitable differential form on $\bigwedge_2^n Y / \bigwedge_1^n Y$.

The first and main objective of this paper is to carry this interpretation to general graded Dirac manifolds, so that both the Hamiltonian and the observables are in equal footing, that is, interpreted as the same kind of object (differential forms of possibly different degree). The usual graded Poisson bracket is defined for Hamiltonian forms of degree between 0 and $n-1$, so we focus on extending this bracket to arbitrary degree, maintaining as much properties as possible. As a result, we obtain a family of possible extensions (dependent on a choice of extensions of the $\sharp$-morphisms) and we shall obtain a bracket

$$\Omega_H^{a-1}(M) \otimes \Omega_H^{b-1}(M) \to \Omega^{a+b-(n+1)}(M),$$

where a and b are no longer confined to $1 \leq a, b \leq n$. The restriction of this bracket to the usual Hamiltonian forms and a particular subclass of Hamiltonian n-forms will yield the usual equations of motion as above.

Regarding the mathematical construction, extensions of Poisson brackets have been studied in symplectic geometry in [41] by Michor and in general Poisson geometry in [32] by Grabowski. It should be noted that the bracket defined in the second cited paper is a modification of the brackets treated in the first. This modification is performed in order to obtain a Jacobi identity (not up to an exact term). We find more suitable for our purposes to generalize the one introduced by Michor, since it is of first order. Nevertheless, both constructions are built upon a generalization of the $\sharp$ mapping, and we follow this strategy closely.

When the extensions of the brackets are finished, we use them to study the field equations and evolution of forms of arbitrary degree, an aspect which was missing in previous versions of the literature. In particular, we would like to highlight the following results:

(i) In Theorem 3.2.2, we build a first extension of the brackets and prove that it still satisfies the main properties of the original one. Afterwards, we give sufficient conditions on the chosen extension of the $\sharp$ mapping for the final extension of the brackets to satisfy the same properties in Theorem 3.3.1.

(ii) In Theorem 4.1.1, we prove that for a given extension of the brackets, the Poisson bracket of a Hamiltonian form $\alpha \in \Omega_H^a(M)$ against the Hamiltonian $\mathcal{H} \in \Omega_H^n(M)[n]$ (see Definition 4.1.2) yields a possible evolution of α, in the sense that there is an Ehresmann connection h that solves the equations determined by $\mathcal{H}$ (equations that do not depend on the extension) and such that $h^*(\mathrm{d}\alpha) = \mathrm{d}\alpha + \{\alpha, \mathcal{H}\}$.

(iii) In Theorem 4.1.2, we prove that there is a one-to-one correspondence between all possible extensions of the brackets to Hamiltonians compatible with the fibered structure and affine maps $\gamma : \{\text{Hamiltonians}\} \to \{\text{connections}\}$ such that $\gamma(\mathcal{H})$ solves the equations determined by $\mathcal{H}$.

(iv) In Theorem 4.2.2, we identify a subfamily of forms, that does not depend on the choice of Hamiltonian, which we call *special Hamiltonian forms*, whose evolution is defined for any solution of the Hamilton–De Donder–Weyl equations. In particular, the bracket against $\mathcal{H}$ does not depend on the choice of extension, making them candidates for being conserved quantities, namely closed forms on solutions (such forms have gained interest in theoretical physics recently, see [29], for instance).

(v) Finally, this subfamily is proved to be a subalgebra under the Poisson bracket in Theorem 4.2.3.

The paper is structured as follows. Section 2 is devoted to study graded Dirac structures on fibered manifolds, and how these structures restrict to submanifolds and quotient manifolds. In Sec. 3, we develop the main ingredients of our study, the possibility of extending the natural brackets on forms to any arbitrary degree. So, in Sec. 4 we can apply the results in the previous section to discuss dynamics on graded Dirac structures on fibered manifolds. In Sec. 5, we show how our theory can be used to study classical field theories by providing two examples: regular Lagrangians and Yang–Mills theories. Finally, Sec. 6 is devoted to summarize the main conclusions and indicate some further work.

Notation and Conventions

We will use these notations and conventions throughout the paper; to facilitate the reading, they are collected here. Furthermore, we indicate where the reader may find the definition of the more technical notation.

(1) All manifolds are assumed C^∞-smooth and finite dimensional.
(2) Einstein's summation convention is assumed throughout the text, unless stated otherwise.
(3) $\bigvee_p M$ denotes the vector bundle of p-vectors on M, $\bigwedge^p TM$.
(4) $\mathfrak{X}^p(M) = \Gamma\big(\bigvee_p M\big)$ denotes the space of all multivector fields of degree p.
(5) $[\cdot,\cdot]$ denotes the Schouten-Nijenhuis bracket on multivector fields. We use the sign conventions of [40].
(6) $\bigwedge^a M$ denotes the vector bundle of a-forms on M, $\bigwedge^a T^*M$.
(7) $\Omega^a(M) = \Gamma\big(\bigwedge^a M\big)$ denotes the space of all a-forms on M.
(8) Given a vector space V, $\mathbb{1}_j$ denotes the identity on $\bigwedge^j V^* \otimes \bigwedge^j V$.
(9) When dealing with multivector fields and differential forms using coordinates, we employ the alternate Kronecker delta notation:

$$\delta^{\mu_1,\ldots,\mu_p}_{\nu_1,\ldots,\nu_p} = \begin{cases} (-1)^\sigma & \text{if } \sigma(\mu_1)\ldots\sigma(\mu_p) = \nu_1\ldots\nu_p, \text{ for some permutation } \sigma, \\ 0 & \text{otherwise.} \end{cases}$$

Furthermore, we employ the multi-index notation, which we will denote as $\widetilde{\mu} = \mu_1,\ldots,\mu_p$. When multi-indices repeat as super-index and sub-index, summation is assumed for *ordered* multi-index. For instance, $\mathrm{d}^a x^{\widetilde{\mu}} \otimes \frac{\partial^a}{\partial^a x^{\widetilde{\mu}}}$ denotes the identity $\mathbb{1}_a \in \bigwedge^a M \otimes \bigvee_a M$, rather than $\frac{1}{a!}\mathrm{d}^a x^{\widetilde{\mu}} \otimes \frac{\partial^a}{\partial^a x^{\widetilde{\mu}}}$, which would be if summation was assumed for all multi-indices.

(10) $\pounds_U \alpha = \mathrm{d}\iota_U \alpha - (-1)^p \iota_U \,\mathrm{d}\alpha$, for $U \in \mathfrak{X}^p(M)$, $\alpha \in \Omega^a(M)$ denotes the Lie derivative along multivector fields. For a proof of its main properties, we refer to [23].

(11) For a subbundle $K \subseteq \bigvee_p M$, and for $a \geq p$, we denote by

$$K^{\circ,a} = \left\{ \alpha \in \bigwedge^a M : \iota_K \alpha = 0 \right\}$$

the annihilator of degree a of K.

(12) Similarly, for a subbundle $S \subseteq \bigwedge^a M$, and $a \geq p$ we denote by

$$S^{\circ,p} = \left\{ U \in \bigvee_p M : \iota_U S = 0 \right\}$$

the annihilator of degree p of S.

(13) For a subbundle $S \subseteq T^*M$, we write $S^{\wedge a} := \bigwedge^a S \subseteq \bigwedge^a M$.

(14) $S^a[j]$ is defined in Eq. (6).

(15) $\Omega_H^{a-1}(M)[j]$ is defined in Eq. (8).

(16) $\{\cdot,\cdot\}_{\widetilde{\sharp}_n}$ is defined in Eq. (10).

2. Graded Dirac Structures on Fibered Manifolds

In this section, we recall the basic definitions and constructions on graded Dirac manifolds (which can be found in [7, 18, 50], for instance). Furthermore, we develop the theory of pullback and pushforwards of these structures which, to our knowledge, was missing in the literature. Lastly, we study all these concepts in the case where a graded Dirac structure on a manifold M is compatible with a fibered structure $\tau \colon M \to X$, which is the setting that we will use to define dynamics.

2.1. *Graded Dirac structures*

Let M be a smooth manifold and suppose that we have a vector subbundle S^n of $\bigwedge^n M$, the bundle of n-forms over M,

$$\begin{tikzcd} S^n \arrow[r, hook] \arrow[d] & \bigwedge^n M \arrow[dl] \\ M & \end{tikzcd},$$

such that the subsequent contractions of S^n by $\mathrm{T}M$,

$$S^{n-1} := \iota_{\mathrm{T}M} S^n, \quad S^{n-2} := \iota_{\mathrm{T}M} S^{n-1}, \dots, S^1 := \iota_{\mathrm{T}M} S^2.$$

define vector subbundles of $\bigwedge^{n-1} M, \bigwedge^{n-2} M, \dots, \mathrm{T}^* M$, respectively. Here we are making a slight abuse of notation denoting

$$\iota_{\mathrm{T}M} S^n = \langle \iota_u \alpha : u \in \mathrm{T}M, \alpha \in S^n \rangle, \quad \iota_{\mathrm{T}M} S^{n-1} = \langle \iota_u \alpha : u \in \mathrm{T}M, \alpha \in S^{n-1} \rangle, \dots$$

Definition 2.1.1 (Hamiltonian form). Let $\mathcal{U}$ be an open subset of M. A **Hamiltonian form** of degree $a-1$ on $\mathcal{U}$, $1 \leq a \leq n$ with respect to the sequence $S^1, \ldots, S^n$ determined by S^n is a differential form $\alpha \in \Omega^{a-1}(M)$ such that $\mathrm{d}\alpha$ takes values in S^a. The space of Hamiltonian forms is denoted as $\Omega_H^{a-1}(\mathcal{U}, S^n)$. If there is no ambiguity, we will just write $\Omega_H^{a-1}(\mathcal{U})$ instead of $\Omega_H^{a-1}(\mathcal{U}, S^n)$.

Let α be a differential form of arbitrary degree. Define

$$\deg_H \alpha := (n-1) - \deg \alpha.$$

We now define our main object of study (see [18] for further details):

Definition 2.1.2 (Graded Poisson bracket). A graded Poisson bracket of order n on M is a bilinear bracket defined for every open subset $\mathcal{U}$ of M,

$$\Omega_H^{a-1}(\mathcal{U}) \otimes \Omega_H^{b-1}(\mathcal{U}) \xrightarrow{\{\cdot,\cdot\}_{\mathcal{U}}} \Omega_H^{a+b-(n+1)}(\mathcal{U})$$

for $1 \leq a, b \leq n$ that satisfies the following properties. For every $\alpha \in \Omega_H^{a-1}(\mathcal{U}), \beta \in \Omega_H^{b-1}(\mathcal{U})$ and $\gamma \in \Omega_H^{c-1}(\mathcal{U})$ we have

(i) *It is graded:*

$$\deg_H \{\alpha, \beta\}_{\mathcal{U}} = \deg_H \alpha + \deg_H \beta;$$

(ii) *It is graded-skew-symmetric:*

$$\{\alpha, \beta\}_{\mathcal{U}} = -(-1)^{\deg_H \alpha \deg_H \beta} \{\beta, \alpha\}_{\mathcal{U}};$$

(iii) *It is local:* If $\mathrm{d}\alpha\,|_x = 0$, then $(\{\alpha, \beta\}_{\mathcal{U}})|_x = 0$;

(iv) *It satisfies a graded Leibniz identity:* Let $\beta^j \in \Omega_H^{b-1}(\mathcal{U}), \gamma_j \in \Omega_H^{c-1}(\mathcal{U})$. If $\beta^j \wedge \mathrm{d}\gamma_j \in \Omega_H^{b+c-1}(\mathcal{U})$, then we have

$$\{\beta^j \wedge \mathrm{d}\gamma_j\,, \alpha\}_{\mathcal{U}} = \{\beta^j, \alpha\}_{\mathcal{U}} \wedge \mathrm{d}\gamma_j + (-1)^{n-\deg_H \beta^j}\, \mathrm{d}\beta^j \wedge \{\gamma_j, \alpha\}_{\mathcal{U}};$$

for $\alpha \in \Omega_H^{n-1}(\mathcal{U})$;

(v) *It is invariant by symmetries:* If $X \in \mathfrak{X}(\mathcal{U})$ and $\pounds_X \alpha = 0$, then $\iota_X \alpha \in \Omega_H^{a-1}(\mathcal{U})$ and

$$\{\iota_X \alpha, \beta\}_{\mathcal{U}} = (-1)^{\deg_H \beta} \iota_X \{\alpha, \beta\}_{\mathcal{U}};$$

(vi) *It satisfies the graded Jacobi identity* (*up to an exact term*)*:*

$$(-1)^{\deg_H \alpha \deg_H \gamma} \{\{\alpha, \beta\}_{\mathcal{U}}, \gamma\}_{\mathcal{U}} + (-1)^{\deg_H \beta \deg_H \alpha} \{\{\beta, \gamma\}_{\mathcal{U}}, \alpha\}_{\mathcal{U}}$$
$$+ (-1)^{\deg_H \gamma \deg_H \beta} \{\{\gamma, \alpha\}_{\mathcal{U}}, \beta\}_{\mathcal{U}} = \text{exact form}.$$

Finally, it satisfies the *compatibility condition*, namely for $\mathcal{V} \subseteq \mathcal{U}$ open, the bracket on $\mathcal{V}$, $\{\cdot,\cdot\}_{\mathcal{V}}$, is the restriction of the bracket on $\mathcal{U}$, $\{\cdot,\cdot\}_{\mathcal{U}}$.

Remark 2.1.1. With the usual grading of differential forms, a graded Poisson bracket of order n can be also understood as a bracket of degree $(1-n)$, in such a way that it is graded once the degrees are modified appropriately.

Remark 2.1.2. The last condition implies that we have a well-defined global bracket

$$\Omega_H^{a-1}(M) \otimes \Omega_H^{b-1}(M) \xrightarrow{\{\cdot,\cdot\}} \Omega_H^{a+b-(n+1)}(M)$$

and, making some abuse of notation, we will denote $\{\cdot,\cdot\}_{\mathcal{U}} = \{\cdot,\cdot\}$. Let us explain why it is necessary to consider the bracket locally. This is due to the fact that $\Omega_H^{a-1}(M)$ could be the space of closed forms, even when S^n is nonzero. For instance, for $n = 1$, if we take the foliation of the torus where every leaf is dense and S^1 is the bundle of forms which are zero when evaluated on tangent vectors to this foliation, then $\mathrm{d}f \in S^1$ only when f is constant.

Actually, the situation described above could still happen for arbitrarily small open subsets $\mathcal{U}$, not due to topology, but due to integrability. Note that in order to have a non-trivial space of Hamiltonian forms (where the trivial one would be defined as the space of closed forms), there must be a non-trivial solution to the following partial differential equation:

$$\mathrm{d}\alpha \in S^a, \quad \alpha \in \Omega^{a-1}(\mathcal{U}),$$

where $\mathrm{d}\alpha \in S^a$ means that $\mathrm{d}\alpha$ takes values in S^a. Hence, if we are not careful, there could be a unique graded Poisson bracket for a given subbundle S^n, the trivial one. The same discussion applies to the graded Leibniz identity and invariance by symmetries. These properties could be meaningless if there are no Hamiltonian forms and vector fields satisfying the requirements. In order to avoid this, we can assume that S^n satisfies the following conditions (the theories to be considered will certainly satisfy these hypotheses):

(i) Locally, for every $1 \leq a \leq n-1$, there exist Hamiltonian forms $\gamma_{ij} \in \Omega_H^{a-1}(\mathcal{U})$, and functions f_i^j such that $S^{a+1} = \langle \mathrm{d}f_i^j \wedge \mathrm{d}\gamma_{ij} \rangle$.
(ii) Locally, for each $1 \leq a \leq n$, there exists a family of Hamiltonian forms γ^j, and a family of vector fields X^i such that $S^a = \langle \mathrm{d}\gamma^j \rangle$, $\pounds_{X^i}\gamma^j = 0$, and $S^{a-1} = \langle \mathrm{d}\iota_{X^i}\gamma^j \rangle$.

These hypotheses, besides guaranteeing that S^a is locally generated by Hamiltonian forms, also guarantee that we can apply freely the Leibniz rule and the invariance by symmetries. In particular, we can prove the following theorem.

Theorem 2.1.1 ([18]). *Let S^n be a vector subbundle of $\bigwedge^n M$ such that $S^{n-1} = \iota_{\mathrm{T}M} S^n, \ldots, S^1 = \iota_{\mathrm{T}M} S^2$ are vector subbundles of $\bigwedge^{n-1} M, \ldots, T^*M$, respectively. Suppose that these subbundles satisfy the hypotheses above. Let $\{\cdot,\cdot\}$ be a graded Poisson bracket on the space of Hamiltonian forms. Then, there exists a unique sequence of vector bundle morphisms, $\sharp_a$ for $1 \leq a \leq n$,*

$$\begin{tikzcd} S^a \arrow[r, "\sharp_a"] \arrow[d] & \bigvee_{n+1-a} M / K_{n+1-a} \arrow[dl] \\ M & \end{tikzcd},$$

where $K_{n+1-a} = \{U \in \bigvee_{n+1-a} M : \iota_U S^{n+1-a} = 0\}$, *satisfying the following properties*:

(i) *The maps* $\natural_a$ *are skew-symmetric, namely*

$$\iota_{\natural_a(\alpha)}\beta = (-1)^{(n+1-a)(n+1-b)}\iota_{\natural_b(\beta)}\alpha$$

for all $\alpha \in S^a$, $\beta \in S^b$.

(ii) *It is integrable: For* $\alpha : M \to S^a$, $\beta : M \to S^b$ *sections such that* $a+b \leq 2n+1$, *and* U, V *multivectors of degree* $p = n+1-a$, $q = n+1-b$, *respectively, such that*

$$\natural_a(\alpha) = U + K_p, \quad \natural_b(\beta) = V + K_q,$$

we have that the $(a+b-n)$*-form*

$$\theta := (-1)^{(p-1)q}\pounds_U\beta + (-1)^q\pounds_V\alpha - \frac{(-1)^q}{2}d(\iota_V\alpha + (-1)^{pq}\iota_U\beta)$$

takes values in S_{a+b-n}, *and*

$$\natural_{a+b-n}(\theta) = [U, V] + K_{p+q-1}.$$

(iii) *For any Hamiltonian forms* $\alpha \in \Omega^{a-1}_H(M)$ *and* $\beta \in \Omega^{b-1}_H(M)$, *we have*

$$\{\alpha, \beta\} = (-1)^{\deg_H \beta}\iota_{\natural_b(\mathrm{d}\beta)}\mathrm{d}\alpha. \tag{1}$$

Conversely, given a sequence of vector bundle maps satisfying the first two previous properties, the bracket defined by Eq. (1) *defines a graded Poisson bracket of order* n.

The previous result induces the following definition:

Definition 2.1.3 (Regular graded Dirac structure of order n). Let M be a smooth manifold, and $S^1, \ldots, S^n$ be a sequence of vector subbundles

$$\begin{tikzcd} S^a \arrow[r, hook] \arrow[d] & \bigwedge^a M \arrow[dl] \\ M & \end{tikzcd}$$

which are related through subsequent contractions, $S^a = \iota_{\mathrm{T}M}S^{a+1}$ (we do not necessarily require them to satisfy the integrability conditions mentioned above). Then, a regular graded Dirac structure of order n on M is a sequence of vector bundle maps

$$\natural_a : S^a \to \bigvee_{n+1-a} M/K_{k+1-a},$$

where K_{n+1-a} is the annihilator of S^{n+1-a} that satisfies the following conditions:

(i) The maps $\natural_a$ are *skew-symmetric*, that is,

$$\iota_{\natural_a(\alpha)}\beta = (-1)^{(n+1-a)(n+1-b)}\iota_{\natural_b(\beta)}\alpha$$

for all $\alpha \in S^a$, $\beta \in S^b$.

(ii) It is *integrable:* For $\alpha : M \to S^a$, $\beta : M \to S^b$ sections such that $a+b \leq 2n+1$, and U, V multivector fields of degree $p = n+1-a$, $q = n+1-b$, respectively such that

$$\sharp_a(\alpha) = U + K_p, \quad \sharp_b(\beta) = V + K_q,$$

we have that the $(a+b-n)$-form

$$\theta := (-1)^{(p-1)q} \pounds_U \beta + (-1)^q \pounds_V \alpha - \frac{(-1)^q}{2} d(\iota_V \alpha + (-1)^{pq} \iota_U \beta)$$

takes values in S_{a+b-n}, and

$$\sharp_{a+b-k}(\theta) = [U, V] + K_{p+q-1}.$$

Remark 2.1.3. Note that a regular graded Dirac structure $(M, S^a, \sharp_a)$ induces a graded Poisson bracket on the space of Hamiltonian forms, by defining $\{\alpha, \beta\} = (-1)^{\deg_H \beta} \iota_{\sharp_b(\mathrm{d}\beta)} \mathrm{d}\alpha$.

Definition 2.1.4 (Flat regular graded Dirac structure). A regular graded Dirac structure of order n will be called **flat** if the sequence of subbundles $S^1, \ldots, S^n$ satisfies the integrability hypotheses (i) and (ii).

Remark 2.1.4. Equivalently, using Theorem 2.1.1, a flat regular graded Dirac structure of order n can be defined by a graded Poisson bracket of order n on the space of Hamiltonian forms $\Omega^{a-1}_H(M)$, for $1 \leq a \leq n$.

Remark 2.1.5. One could weaken the hypotheses of the objects in Definition 2.1.3, and work instead with vector subbundles of

$$\bigvee_p M \oplus_M \bigwedge^{n+1-p} M,$$

as in [7, 18, 47, 48], corresponding to the image of the vector bundle morphism

$$D^p = \{(u, \alpha) : \alpha \in S^{n+1-p}, \sharp_{n+1-p}(\alpha) = u + K_p\},$$

see [18, Proposition III A.2]. We prefer to work with regular graded Dirac structures because of its immediate connection to brackets, which is our main focus in this paper. Nevertheless, we expect that the results presented here may be extended to the general case with no difficulty.

Remark 2.1.6. The case $n = 1$ recovers the usual definition of Dirac structure. Indeed, a regular "graded" Dirac structure of order one on M (following Definition 2.1.3) would correspond to a map

$$\sharp : S \to \mathrm{T}M/K,$$

where $S \subseteq \mathrm{T}^* M$ is a vector subbundle. Integrability implies that K is an integrable distribution in the sense of Frobenius. Then, both skew-symmetry and integrability of $\sharp$ translate into

$$D := \{(v, \alpha) \in \mathrm{T}M \oplus_M T^*M : \sharp(\alpha) = v + K\} \subset \mathrm{T}M \oplus_M T^*M$$

defining a Dirac structure in the usual case present in the literature (see [6, 15, 21, 22]). Of course, this does not recover all possible Dirac structures, but only those $D \subset \mathrm{T}M \oplus_M T^*M$ whose projection onto T^*M defines a vector subbundle. In any case, we can recover all possible Dirac structures if we work with the notion presented in Remark 2.1.5 instead.

Remark 2.1.7. An important feature of these structures is that they are completely determined by the structure on degree n, since the vector bundle map $\sharp_n\colon S^n \to \mathrm{T}M/K_1$ determines the rest defining $\sharp_a(\iota_U\alpha) := \sharp_n(\alpha) \wedge U + K_{n+1-a}$, where $\alpha \in S^n$ and $U \in \bigvee_{n-a} M$ (see [18]).

2.2. *Restrictions and quotients of graded Dirac structures*

Now we will study induced graded Dirac structures by restrictions to sub-manifolds and projections to a quotient. More generally, we will define the *pullback* and *pushforward* of a graded Dirac structure by a smooth map (see [8] for the definitions in Dirac geometry).

Let N, M and P be smooth manifolds, and let M be endowed with a regular graded Dirac structure $(S^a, \sharp_a)$. Suppose we had an immersion $i : N \to M$ and a submersion $\pi : M \to P$. One should define the induced structures by i and π on N and P, respectively, as maximal regular Dirac structures such that both i and π preserve them. The preservation of the structure is given by the notion of a *graded Dirac map*. In Dirac geometry (in some contexts called generalized geometry, see [34]), there are two notions, forward Dirac maps (generalizing pushforward of multivectors) and backward Dirac maps (generalizing pullbacks of forms), see [8]. Let us first define a general notion of graded Dirac map in order to define the generalizations of the previous concepts.

Definition 2.2.1. Let $(N, \widetilde{S}^a, \widetilde{\sharp}_a)$ and $(M, S^a, \sharp_a)$ be two regular graded Dirac manifolds and $f\colon N \to M$ be a smooth map. Then, f is said to be a graded Dirac map if the following diagram is commutative for every $x \in N$ and every $a = 1, \dots, n$:

$$\begin{tikzcd}
(f^*)^{-1}\left(\widetilde{S}^a\big|_x\right) \bigcap S^a\big|_{f(x)} \arrow[r, "f^*"] \arrow[d, "\sharp_a"] & \widetilde{S}^a\big|_x \arrow[d, dashed, "\widetilde{\sharp}_a"] \\
\bigwedge^{n+1-a}\left(\mathrm{T}_{f(x)} M\right) / \left(K_{n+1-a} + f_*(\widetilde{K}_{n+1-a})\right) & \bigwedge^{n+1-a}(\mathrm{T}_x N) / \widetilde{K}_{n+1-a} \arrow[l, "f_*"]
\end{tikzcd} .$$

Some clarification on the notation is due. First, f induces a map $f^*\colon \bigwedge^a T^*_{f(x)}M \to \bigwedge^a T^*_x N$, so $(f^*)^{-1}(\widetilde{S}^a\big|_x)$ denotes the pre-image of $\widetilde{S}^a\big|_x \subseteq \bigwedge^a T^*_x N$ by the pullback map. Finally, making some abuse of notation, we interpret the map on the left $\sharp_a$ as the composition

$$S^a \xrightarrow{\sharp_a} \bigvee_{n+1-a} M/K_{n+1-a} \to \bigvee_{n+1-a} M/(K_{n+1-a} + f_*(\widetilde{K}_{n+1-a})),$$

where the second map is the projection onto the quotient. Hence, a smooth map $f\colon N \to M$ is said to be a graded Dirac map if it satisfies $\sharp_a(\alpha|_{f(x)}) = f_*(\widetilde{\sharp}_a(f^*\alpha))$ modulo $K_{n+1-a} + f_*(\widetilde{K}_{n+1-a})$ for every $\alpha \in S^a|_{f(x)}$ such that $f^*\alpha \in \widetilde{S}^a$.

That Definition 2.2.1 is the correct notion for a structure-preserving map between graded Dirac manifolds, motivated by the following property.

Proposition 2.2.1. *Let $(N, \widetilde{S}^a, \widetilde{\sharp}_a)$ and $(M, S^a, \sharp_a)$ be two regular graded Dirac manifolds of order n and $f : N \to M$ be a graded Dirac map. Let α and β be Hamiltonian forms on M such that both $f^*\alpha$ and $f^*\beta$ are Hamiltonian on N. Then,*

$$\{f^*\alpha, f^*\beta\} = f^*\{\alpha, \beta\}.$$

Proof. Suppose $\beta \in \Omega_H^{b-1}(M)$. Then, we have

$$\begin{aligned} f^*\{\alpha,\beta\} &= (-1)^{\deg_H \beta} f^*(\iota_{\sharp_b(\mathrm{d}\beta)}\mathrm{d}\alpha) \\ &= (-1)^{\deg_H \beta} f^*(\iota_{f_*\sharp_b(\mathrm{d}f^*\beta)}\mathrm{d}\alpha) = (-1)^{\deg_H \beta}\iota_{\sharp_b(\mathrm{d}f^*\beta)} f^*\mathrm{d}\alpha = \{f^*\alpha, f^*\beta\}, \end{aligned}$$

which finishes the proof. □

Furthermore, in the case that f is *well behaved* with respect to both graded Dirac structures, Definition 2.2.1 is not only sufficient but also necessary for a map $f : N \to M$ to preserve brackets in the sense of Proposition 2.2.1.

Proposition 2.2.2. *Let $(N, \widetilde{S}^a, \widetilde{\sharp}_a)$ and $(M, S^a, \sharp_a)$ be two regular graded Dirac manifolds of order n and $f : N \to M$ be a smooth map such that for every $\alpha \in \Omega_H^{a-1}(N)$ there is $\gamma \in \Omega_H^{a-1}(M)$ with $f^*\gamma = \alpha$. Then, f is a graded Dirac map if and only if it satisfies*

$$\{f^*\alpha, f^*\beta\} = f^*\{\alpha, \beta\}$$

for every pair of Hamiltonian forms α and β on M such that both $f^*\alpha$ and $f^*\beta$ are Hamiltonian forms on N.

Proof. Necessity follows from Proposition 2.2.1. Now, for sufficiency, let $\alpha \in (f^*)^{-1}(\widetilde{S}^a) \cap S^a$. We need to show that $\sharp_a(\alpha) = f_*\sharp_a(f^*\alpha) \mod K_{n+1-a} + f_*(\widetilde{K}_{n+1-a})$. Now, since

$$\begin{aligned} &((f^*)^{-1}(\widetilde{S}^{n+1-a}) \cap S^{n+1-a})^{\circ, n+1-a} \\ &\quad = ((f^*)^{-1}(\widetilde{S}^{n+1-a}))^{\circ, n+1-a} + (S^{n+1-a})^{\circ, n+1-a} \\ &\quad = K_{n+1-a} + f_*(\widetilde{K}_{n+1-a}), \end{aligned}$$

it is enough to show that $\iota_{\sharp_a(\alpha)}\beta = \iota_{f_*\sharp_a(f^*\alpha)}\beta$ for every $\beta \in (f^*)^{-1}(\widetilde{S}^{n+1-a}) \cap S^{n+1-a}$, which clearly follows from the bracket preservation hypothesis. $\square$

Now we can define the desired induced graded Dirac structures as maximal structures (with respect to inclusion and extension of the corresponding $\sharp$ morphisms) on N and P such that both i and π are graded Dirac maps. These are characterized by two particular kind of maps, those that we call *backward* and *forward*, generalizing the well-known concepts in Dirac geometry [9]:

Definition 2.2.2 (Backward and forward maps). A graded Dirac map $f : N \to M$ is said to be backward if $\widetilde{S}^a|_x = \{f^*\alpha : \alpha \in S^a|_{f(x)}$ and $\sharp_a(\alpha)$ is tangent to f, modulo $K_{n+1-a}\}$, and it is said to be forward if $S^a|_{f(x)} = (f^*)^{-1}(\widetilde{S}^a|_x)$ for every $a = 1, \ldots, n$ and $x \in N$.

Definition 2.2.3 (Pullback and pushforward of graded Dirac structures). Let $(M, S^a, \sharp_a)$ be a regular graded Dirac manifold of order n and $f : N \to M$ be a smooth map (respectively, $\pi : M \to P$, be a submersion). A graded Dirac structure on N, $(\widetilde{S}^a, \widetilde{\sharp}_a)$ (respectively, on P) is called the pullback of $(S^a, \sharp_a)$ (respectively, the pushforward) if f is a backward (repectively, forward) graded Dirac map between these two structures.

Some elementary properties are in order.

Theorem 2.2.1. *When they exist, both pullback and pushforward of graded Dirac structures are unique.*

Proof. Let $(M, S^a, \sharp_a)$ be a graded Dirac manifold, $f : N \to M$ be a smooth map and $\pi : M \to P$ be a submersion. Clearly, if there are graded Dirac structures on N and P such that f is a backward graded Dirac map and π is a forward graded Dirac map, the corresponding subbundles $\widetilde{S}^a$ are unique, by definition. Therefore, it is sufficient to check that the choice of the $\sharp$ maps is unique as well. Let us study both cases separately:

(i) Pullbacks: Note that $(f^*)^{-1}(\widetilde{S}^a) \cap S^a \xrightarrow{f^*} \widetilde{S}^a$ is surjective. Suppose there were two maps

$$\widetilde{\sharp}_a, \widetilde{\sharp}'_a : \widetilde{S}^a \to \bigvee_{n+1-a} N/\widetilde{K}_{n+1-a}$$

such that $f_*(\widetilde{\sharp}_a(f^*\alpha)) = f_*(\widetilde{\sharp}'_a(f^*\alpha)) = \sharp_a(\alpha)$ modulo $K_{n+1-a} + f_*\widetilde{K}_{n+1-a}$ for every $\alpha \in (f^*)^{-1}(\widetilde{S}^a) \cap S^a$. Then, for every $\alpha \in S^a$, we would have

$$f_*(\widetilde{\sharp}_a(f^*\alpha)) - f_*(\widetilde{\sharp}'_a(f^*\alpha))$$
$$\in (K_{n+1-a} + f_*\widetilde{K}_{n+1-a}) \cap f_*\left(\bigvee_{n+1-a} N\right) \subseteq f_*\widetilde{K}_{n+1-a},$$

so that $f_*(\widetilde{\sharp}_a(f^*\alpha) - \sharp'_a(f^*\alpha)) \in f_*(\widetilde{K}_{n+1-a})$. Hence, $\sharp_a(f^*\alpha) - \sharp'_a(f^*\alpha) \in (f_*)^{-1}(f_*(\widetilde{K}_{n+1-a}))$. Since this last subspace is easily seen to be $\widetilde{K}_{n+1-a}$, we have $\sharp_a(f^*\alpha) - \sharp'_a(f^*\alpha) \in \widetilde{K}_{n+1-a}$, finishing the proof.

(ii) Pushforwards: Uniqueness in the case of pushforwards follows from the observation that $\pi_*(K_{n+1-a}) \subseteq \widetilde{K}_{n+1-a}$, given that π^* is injective (since π is a submersion). Therefore, if π is a forward graded Dirac map, we necessarily have a commutative diagram

$$\begin{tikzcd} (\pi^*)^{-1}(S^a) \arrow[r, "\pi^*"] \arrow[d, "\widetilde{\sharp}_a"'] & S^a \arrow[d, "\sharp_a"] \\ \bigvee_{n+1-a} P/\widetilde{K}_{n+1-a} & \bigvee_{n+1-a} M/K_{n+1-a} \arrow[l, "\pi_*"] \end{tikzcd},$$

from which uniqueness follows. □

Remark 2.2.1. Actually, from the analysis done in Theorem 2.2.1 we can explicitly compute the pullback and pushforward of regular graded Dirac structures. This will be useful in the sequel.

We now turn onto the problem of determining the conditions for existence of both pullback and pushforward of graded Dirac structures. The matter of existence of pullbacks is purely a matter of regularity:

Proposition 2.2.3. *Let $f : N \to M$ be a smooth map and let M be endowed with a regular graded Dirac structure $(S^a, \sharp_a)$. Suppose*

$$\widetilde{S}^a := \{f^*\alpha : \sharp_a(\alpha) \text{ is tangent to } f\}$$

defines a vector subbundle of $\bigwedge^a N$ for every $1 \leq a \leq n$. Then, there exists the pullback of the graded Dirac structure by f.

Proof. Skew symmetry of the maps follows easily from the definition. Integrability follows from the naturality of the Schouten–Nijenhuis bracket. □

In particular, if an embedding $i : N \to M$ is well behaved with respect to the graded Dirac structure on M, then we can restrict said structure to N. However, the existence of pushforwards is only guaranteed under the following hypotheses:

Proposition 2.2.4. *Let $\pi : M \to P$ be a submersion and let M be endowed with a graded Dirac structure $(S^a, \sharp_a)$. Then, the pushforward of $(S^a, \sharp_a)$ exists if and only if for each pair of points x and $y \in M$ such that $\pi(x) = \pi(y)$ we have*

(i) $\{\alpha \in \bigwedge^a \mathrm{T}^*_{\pi(x)}P : (\mathrm{d}_x\pi)^*\alpha \in S^a|_x\} = \{\alpha \in \bigwedge^a \mathrm{T}^*_{\pi(x)}P : (\mathrm{d}_y\pi)^*\alpha \in S^a|_y\}$,

(ii) *Given $\alpha \in \bigwedge^a \mathrm{T}^*_{\pi(x)}P$ such that $(\mathrm{d}_x\pi)^*\alpha \in S^a|_x$ and $(\mathrm{d}_\pi)^*\alpha \in S^a|_y$, then*

$$(\mathrm{d}_x\pi)_*\sharp_a((\mathrm{d}_x\pi)^*\alpha) = (\mathrm{d}_y\pi)_*\sharp_a((\mathrm{d}_y\pi)^*\alpha).$$

Proof. Clearly, the previous hypotheses allow us to define induced maps $\widetilde{\sharp}_a : \widetilde{S}^a \to \bigvee_{n+1-a} P/\widetilde{K}_{n+1-a}$ that are skew-symmetric. Integrability follows from the naturality of the Lie derivative and Schouten–Nijenhuis bracket. $\square$

Such hypotheses are guaranteed, for instance, when dealing with a quotient by a Lie group action that preserves the graded Dirac structure, so that the space of orbits inherits a natural graded Dirac structure, namely the pushforward of Proposition 2.2.4. However, an in-depth study of reduction is not the objective of this paper.

2.3. *Fibered graded Dirac manifolds*

Now, we shall focus on describing graded Dirac structures compatible with a fibered manifold $\tau : M \to X$, which is the structure that we use to describe classical field theories.

Let $\tau : M \to X$ be a fibered manifold, namely, a surjective submersion. Let us first introduce some bit of terminology that we will use constantly throughout our study:

Definition 2.3.1 (Horizontal, basic, semi-basic forms). A differential $(a-1)$-form $\alpha \in \Omega^{a-1}(M)$, for $a \geq 1$, is called:

(i) Semi-basic over X with respect to the projection $\tau : M \to X$ if $\iota_v \alpha = 0$ for every vector $v \in \mathrm{T}M$ such that $\mathrm{d}\tau \cdot v = 0$.
(ii) Basic over X with respect to the projection $\tau : M \to X$ if $\alpha = \tau^* \beta$, for certain a-form β (which will be unique) on X.
(iii) r-horizontal over X with respect to the projection $\tau : M \to X$, for $0 \leq r \leq a$ if $\iota_{v_1,\ldots,v_{a-r}} \alpha = 0$ for every sequence of vectors $v_1, \ldots, v_{a+1-r} \in \mathrm{T}M$ such that $\mathrm{d}\tau \cdot v_i = 0$.

Remark 2.3.1. Note that the notion of semi-basic $(a-1)$-form and $(a-1)$-horizontal a-form coincide.

Definition 2.3.2 (Fibered graded Dirac manifolds). Let $\tau\colon M \to X$ be a fibered manifold over an n-dimensional manifold X. A graded Dirac structure on M of order n, $(S^a, \sharp_a)$ is called fibered over X if the following two conditions hold:

(i) S^a consists of $(a-1)$-horizontal forms over X.
(ii) Every semi-basic a-form over X is in S^a.

Remark 2.3.2. Let us motivate this definition. First, as we will see in Theorem 4.2.1, Hamiltonian forms on a graded Dirac manifold will be the forms that are candidates to have defined evolution. Since we want this evolution to be linear on the future Hamiltonian, we ask for the a-forms to be $(a-1)$-horizontal (or, equivalently, at most 1-vertical). Second, it is only natural to ask for basic forms to have defined evolution. This issue is represented by the second condition.

We now proceed to study the relationship of fibered structures to the previous constructions of graded Dirac maps, pullbacks and pushforwards.

Proposition 2.3.1. *Let $\tau_1\colon N \to X$, $\tau_2\colon P \to X$ and $\tau\colon M \to X$ be fibered manifolds and let $\tau\colon M \to X$ be endowed with a fibered graded Dirac structure. Then, both pullback and pushforwards (if they are well defined) by smooth maps $f : N \to M$ and submersions $\pi\colon M \to P$, respectively, are fibered when f and π are fibered maps.*

Proof. Both are easily checked from the definition of backward and forward graded Dirac maps. $\square$

Let us study an example which will be of relevance for our purposes:

Example 2.3.1. Let $\pi\colon Y \to X$ be a fibered manifold over an n-dimensional base, X, and let $\widetilde{M} := \bigwedge_2^n Y \subseteq \bigwedge^n Y$ denote the vector subbundle of $(n-1)$-horizontal forms, that is, the space consisting of those $\alpha \in \bigwedge_2^n Y$ such that $\iota_{e_1 \wedge e_2}\alpha = 0$ for every $e_i \in \mathrm{T}Y$ with $\mathrm{d}\pi(e_i) = 0$. If (x^μ, y^i) denote fibered coordinates on Y so that $\pi(x^\mu, y^i) = (x^\mu)$, the previous vector bundle is generated by the forms

$$\mathrm{d}^n x, \quad \mathrm{d}y^i \wedge \mathrm{d}^{n-1}x_\mu,$$

where $\mathrm{d}^n x$ and $\mathrm{d}^{n-1}x_\mu$ denote $\mathrm{d}x^1 \wedge \cdots \wedge \mathrm{d}x^n$ and $\iota_{\frac{\partial}{\partial x^\mu}} \mathrm{d}^n x$, respectively. Therefore, the subbundle $\bigwedge_2^n Y$ admits natural coordinates (x^μ, y^i, p, p_i^μ) representing the form $\alpha = p\mathrm{d}^n x + p_i^\mu \mathrm{d}y^i \wedge \mathrm{d}^{n-1}x_\mu$. There is a canonical form, called the Liouville form (or the canonical multisymplectic potential)

$$\Theta = p\mathrm{d}^n x + p_i^\mu \mathrm{d}y^i \wedge \mathrm{d}^{n-1}x_\mu,$$

which induces a canonical multisymplectic form on this manifold, which is given by the closed non-degenerate form

$$\Omega = -\mathrm{d}\Theta = -\mathrm{d}p \wedge \mathrm{d}^n x - \mathrm{d}p_i^\mu \wedge \mathrm{d}y^i \wedge \mathrm{d}^{n-1}x_\mu.$$

This closed form induces a map $\flat : \mathrm{T}\widetilde{M} \to \bigwedge^n \widetilde{M}$ given by $\flat(v) := \iota_v\Omega$. This map defines a monomorphism, so that its inverse $\sharp_n := \flat^{-1}$ is a well-defined map between $S^n := \operatorname{Im}\flat$ and $\mathrm{T}M$:

$$\sharp_n\colon S^n \to \mathrm{T}M, \quad \sharp_n(\alpha) = \flat^{-1}(\alpha).$$

In coordinates, this map reads as follows

$$\sharp_n(\mathrm{d}^n x) = -\frac{\partial}{\partial p}, \quad \sharp_n(\mathrm{d}y^i \wedge \mathrm{d}^{n-1}x_\mu) = -\frac{\partial}{\partial p_i^\mu},$$

$$\sharp_n(\mathrm{d}p_i^\mu \wedge \mathrm{d}^{n-1}x_\mu) = \frac{\partial}{\partial y^i}, \quad \sharp_n(\mathrm{d}p \wedge \mathrm{d}^{n-1}x_\nu - \mathrm{d}p_i^\mu \wedge \mathrm{d}y^i \wedge \mathrm{d}^{n-1}x_{\mu\nu}) = \frac{\partial}{\partial x^\nu}.$$

The reader may check that this is a graded Dirac structure but, as we can see, it is not fibered over X. This setting (usually done via the form Ω) is called the *extended* covariant formalism of classical field theory. Then, there is the *reduced* formalism, which takes place in the manifold obtained by quotienting $\widetilde{M}$ by semi-basic forms, $\bigwedge_1^n Y$ so that the manifold is $M := \bigwedge_2^n Y \big/ \bigwedge_1^n Y$ with induced coordinates (x^μ, y^i, p_i^μ). When working in the reduced formalism, the reader may find one of two options in the literature:

(i) The first option, given a section (representing a Hamiltonian) $h : M \to \widetilde{M}$, to work with the multisymplectic structure induced by $\Omega_h = h^*\Omega$, see [13, 43], for instance.
(ii) The second option, to identify those basic Hamiltonian forms (with respect to the projection $\widetilde{M} \to M$) whose bracket is still basic, and work with the reduced bracket on M, see [2, 14], among others.

The first method is uncomfortable if one aims to develop a bracket theory independent of the Hamiltonian, so we will work with the second [14]. This actually corresponds to our notion of pushforward of a graded Dirac structure. The induced structure is easily seen to be $S^n = \langle \mathrm{d}^n x, \mathrm{d}y^i \wedge \mathrm{d}^{n-1}x_\mu, \mathrm{d}p_i^\mu \wedge \mathrm{d}^{n-1}x_\mu \rangle$ with $\sharp_n$ given by

$$\sharp_n(\mathrm{d}^n x) = 0, \quad \sharp_n(\mathrm{d}y^i \wedge \mathrm{d}^{n-1}x_\mu) = -\frac{\partial}{\partial p_i^\mu}, \quad \sharp_n(\mathrm{d}p_i^\mu \wedge \mathrm{d}^{n-1}x_\mu) = \frac{\partial}{\partial y^i}.$$

We immediately see that this is a fibered graded Dirac manifold over X.

Remark 2.3.3. In the extended formalism, there are several and equivalent ways of determining the dynamics, see [49] for a recent treatment of the subject.

3. Extensions of Graded Poisson Brackets

We now turn to the main objective of our study: defining extensions of graded Poisson brackets on a regular graded Dirac manifold. More particularly, we are interested in finding a bracket

$$\Omega_H^{a-1}(M) \otimes \Omega_H^{b-1}(M) \to \Omega_H^{a+b-(n+1)}(M),$$

that satisfies as much properties of Definition 2.1.2 as possible, where $a, b \geq 1$ are now unbounded. As we will immediately see, this bracket will depend upon certain choices. Nevertheless, there will be pairs of forms such that the bracket is independent of all choices made. These pairs will then be used to define dynamics.

We will use the techniques employed in [32, 41], that is, we start by defining a suitable extension of the $\sharp$ mappings to a larger family of forms, and then the brackets will be defined by Eq. (1) up to a constant, namely

$$\{\alpha, \beta\} := (-1)^{\deg_H \beta} C \iota_{\sharp(\mathrm{d}\beta)} \mathrm{d}\alpha,$$

where α and β are suitable forms of arbitrary degree and $\sharp(\mathrm{d}\beta)$ is the aforementioned extension. Our study is divided in three steps:

(i) First step: In order to motivate the construction in the two subsequent steps, we show that the maps

$$\sharp_a : S^a \to \bigvee_{n+1-a} M/K_{n+1-a}$$

for $1 \leq a \leq n$ may be extended to maps $\sharp_a : S^b \to S^{b-a} \otimes \bigvee_{n+1-a} M/K_{n+1-a}$. Furthermore, one may define the brackets using any possible extension, since we will show that there is a constant $C_{a,b}$ such that $\iota_{\sharp_b(\beta)}\gamma = C_{a,b} \cdot \iota_{\sharp_a(\beta)}\gamma$ for every $\beta \in S^b$, $b \geq a$ and $\gamma \in S^c$ with $c \geq n+1-a$.

(ii) Second step: We show how to extend the first $\sharp$-mapping

$$\sharp_1 \colon S^1 \to \bigvee_n M/K_n$$

to a map $\widetilde{\sharp}_1 : (S^1)^{\wedge a} \to \bigwedge^{a-1} M \otimes \bigvee_n M/K_n$, where

$$(S^1)^{\wedge a} := \bigwedge^a S^1 \hookrightarrow \bigwedge^a M,$$

in a canonical way, for arbitrary a. This extension will allow us to introduce a bracket of Hamiltonian $(n-1)$-forms and arbitrary forms Θ with $\mathrm{d}\Theta \in (S^1)^{\wedge a}$ using Eq. (1). This is the bracket that we will use to define dynamics.

(iii) Last step: We deal with the issue of extending the rest of the $\sharp$-mappings to forms of arbitrary degree. Compatible extensions of $\widetilde{\sharp}_j$ with the map $\widetilde{\sharp}_1$ presented in the second step will only exist for a particular subfamily of forms, those whose exterior derivative takes values in a particular subbundle which we denote by $S^a[j]$. Thus, we will look for extensions $\widetilde{\sharp}_j : S^a[j] \to \bigwedge^{a-j} M \otimes \bigvee_{n+1-j} M/K_j$, and study their main properties.

3.1. *First step*: *Extension of $\sharp_a$ to S^b, for $n \geq b \geq a$*

Just as a motivation, let us first notice that we may extend the mapping

$$\sharp_a : S^a \to \bigvee_{n+1-a} M/K_{n+1-a}$$

to every subbundle S^b, with $n \geq b \geq a$, but first, let us briefly explain a very useful technique that we will employ throughout the paper:

Remark 3.1.1. Throughout the rest of the text, we will exploit the fact that given a linear map between vector spaces $f \colon V_1 \otimes V_3 \to V_2$ there is a unique map

$$g \colon V_1 \to V_2 \otimes V_3^*$$

such that, for every $v_1 \in V_1$ and $v_3 \in V_3$, we have $\langle g(v_1), v_3\rangle = f(v_1 \otimes v_3)$, where $\langle \cdot, \cdot \rangle \colon (V_2 \otimes V_3^*) \times V_3 \to V_2$ denotes the natural pairing. Indeed, g is simply the adjoint of the map f, taken on the second component of the tensor product. In most cases, V_3 will be a space of forms, e.g. $V_3 = S^a$, and we will identify

$$(S^a)^* \cong \bigvee_{n+1-a} M/K_{n+1-a},$$

where the pairing is given by contraction. We will refer to this process as *taking adjoints.*

Proposition 3.1.1. *For $1 \leq a \leq n$ and $a \leq b \leq n$ there is a unique extension*

$$\sharp_a : S^b \to S^{b-a} \otimes \bigvee_{n+1-a} M/K_{n+1-a},$$

such that, if $\beta \in S^b$ and $\gamma \in S^c$ with $c \geq n+1-a$, then we have

$$\iota_{\sharp_a(\beta)}\gamma = \binom{b+c-(n+1)}{b-a}\iota_{\sharp_b(\beta)}\gamma = \frac{(b+c-(n+1))!}{(b-a)!(a+c-(n+1))!}\iota_{\sharp_b(\beta)}\gamma. \tag{2}$$

Here, contraction by a multivector valued form is defined as $\iota_{\theta\otimes u}\gamma := \theta \wedge \iota_u\gamma$, generalizing the usual contraction by vector valued forms (see [32]).

Proof. Particularizing Eq. (2) for $\gamma \in S^{n+1-a}$, we must have $\iota_{\sharp_a(\beta)}\gamma = \iota_{\sharp_b(\beta)}\gamma$. The second term defines a linear mapping

$$S^b \otimes S^{n+1-a} \to S^{b-a}.$$

Taking the adjoint of the previous mapping (see Remark 3.1.1), we see that there exists a unique map

$$\sharp_a : S^b \to S^{b-a} \otimes \bigvee_{n+1-a} M/K_{n+1-a}$$

that satisfies Eq. (2) for $c = n+1-a$. It only remains to check that such a map will also satisfy it for arbitrary $n \geq c \geq n+1-a$. This will be a consequence of the following lemmas:

Lemma 3.1.1. *Let V be a finite dimensional vector space and $u \in \bigwedge^p V$. For $a \geq p$ we have an induced map*

$$\bigwedge^a V^* \to \bigwedge^{a-p} V^*, \quad \alpha \mapsto \iota_u\alpha.$$

Then, interpreting this linear map as an element $C_u \in \bigwedge^{a-p} V^* \otimes \bigwedge^a V$, we have

$$C_u = (-1)^{p(a-p)}\mathbb{1}_{a-p} \wedge u = u \wedge \mathbb{1}_{a-p},$$

where $\mathbb{1}_{a-p} \in \bigwedge^{a-p} V^* \otimes \bigwedge^{a-p} V$ denotes the identity, and the exterior product is given by $(\theta \otimes v) \wedge u = \theta \otimes (v \wedge u)$ or $u \wedge (\theta \otimes v) = \theta \otimes (u \wedge v)$.

Proof. Let $\{e_i\}$ denote a basis of V and suppose $u = u^{i_1\dots i_p} e_{i_1} \wedge \cdots \wedge e_{i_p}$. Then,

$$\iota_u e^{j_1} \wedge \cdots \wedge e^{j_a} = \delta^{j_1\dots j_a}_{i_1\dots i_p k_{p+1}\dots k_a} u^{i_1\dots i_p} e^{k_{p+1}} \wedge \cdots \wedge e^{k_a}.$$

Hence, we have

$$C_u = \delta^{j_1\dots j_a}_{i_1\dots i_p k_{p+1}\dots k_a} u^{i_1\dots i_p} e^{k_{p+1}} \wedge \cdots \wedge e^{k_a} \otimes e_{j_1} \wedge \cdots \wedge e_{j_a}.$$

Furthermore, $\mathbb{1}_{a-p} = e^{k_{p+1}} \wedge \cdots \wedge e^{k_a} \otimes e_{k_{p+1}} \wedge \cdots \wedge e_{k_a}$ and we have

$$\begin{aligned}
\mathbb{1}_{a-p} \wedge u &= u^{i_1\dots i_p} e^{k_{p+1}} \wedge \cdots \wedge e^{k_a} \otimes e_{k_{p+1}} \wedge \cdots \wedge e_{k_a} \wedge e_{i_1} \wedge \cdots \wedge e_{i_p} \\
&= \delta^{j_1\dots j_a}_{k_{p+1}\dots k_a i_1\dots i_p} u^{i_1\dots i_p} e^{k_{p+1}} \wedge \cdots \wedge e^{k_a} \otimes e_{j_1} \wedge \cdots \wedge e_{j_a} \\
&= (-1)^{p(a-p)} \delta^{j_1\dots j_a}_{i_1\dots i_p k_{p+1}\dots k_a} u^{i_1\dots i_p} e^{k_{p+1}} \wedge \cdots \wedge e^{k_a} \otimes e_{j_1} \wedge \cdots \wedge e_{j_a},
\end{aligned}$$

where in the last equality we have switched $k_{p+1}\dots k_a i_1 \dots i_p$ to $i_1\dots i_p k_{p+1}\dots k_a$. Comparing both expressions, we conclude that $C_u = (-1)^{p(a-p)} \mathbb{1}_{a-p} \wedge u$, finishing the proof. $\square$

Lemma 3.1.2. *Let $\mathbb{1}_a \in \bigwedge^a V^* \otimes \bigwedge^a V$ denote the identity. Then, for $b \geq a$, and $\beta \in \bigwedge^b V^*$, we have*

$$\iota_{\mathbb{1}_a} \beta = \binom{b}{a} \beta = \frac{b!}{a!(b-a)!} \beta.$$

Proof. Again, let $\{e_i\}$ denote a basis of V and suppose $\beta = \beta_{i_1\dots i_b} e^{i_1} \wedge \cdots e^{i_b}$. Now, since

$$\mathbb{1}_a = e^{j_1} \wedge \cdots \wedge e^{j_a} \otimes e_{j_1} \wedge \cdots \wedge e_{j_a}$$

we have

$$\begin{aligned}
\iota_{\mathbb{1}_a} \beta &= \beta_{i_1\dots i_b} e^{j_1} \wedge \cdots \wedge e^{j_a} \wedge \left(\iota_{e_{j_1} \wedge \cdots \wedge e_{j_a}} e^{i_1} \wedge \cdots \wedge e^{i_b} \right) \\
&= \delta^{i_1\dots i_b}_{j_1\dots j_a k_{a+1}\dots k_b} \beta_{i_1\dots i_b} e^{j_1} \wedge \cdots \wedge e^{j_a} \wedge e^{k_{a+1}} \wedge \cdots \wedge e^{k_b} \\
&= \binom{b}{a} \beta_{i_1\dots i_b} e^{i_1} \wedge \cdots \wedge e^{i_b} = \binom{b}{a} \beta,
\end{aligned}$$

which finishes the proof. $\square$

Using Lemma 3.1.1 it follows that for $\beta \in S^b$ with $b \geq a$, we may write

$$\sharp_a(\beta) = \sharp_b(\beta) \wedge \mathbb{1}_{b-a}.$$

In general, notice that for a multivector valued form $\theta \otimes v \in \bigwedge^a V^* \otimes \bigwedge^q V$, and $u \in \bigwedge^p V$ we have that

$$\iota_{u \wedge (\theta \otimes v)} \gamma = \iota_{\theta \otimes (u \wedge v)} \gamma = \theta \wedge \iota_{u \wedge v} \gamma = \theta \wedge \iota_v \iota_u \gamma = \iota_{\theta \otimes v} \iota_u \gamma$$

for every $\gamma \in \bigwedge^c V^*$. Hence, for $\gamma \in S^c$, with $n \geq c \geq n+1-a$,

$$\iota_{\sharp_a(\beta)} \gamma = \iota_{\mathbb{1}_{b-a}} \iota_{\sharp_b(\beta)} \gamma.$$

Now, since $c \geq n+1-a$, we have $c-(n+1-b) \geq b-a$ and, using Lemma 3.1.2, we conclude that

$$\iota_{\sharp_a(\beta)} \gamma = \binom{b+c-(n+1)}{b-a} \iota_{\sharp_b(\beta)} \gamma,$$

which finishes the proof of Proposition 3.1.1.

Remark 3.1.2. This implies that in the definition of graded Poisson brackets we can use any $\sharp$ mapping, as long as we are careful with the degrees of the forms to be considered and the coefficients appearing in Proposition 3.1.1. Indeed, we know that

$$\begin{aligned} \{\gamma, \beta\} &= (-1)^{\deg_H \beta} \binom{b+c-(n+1)}{b-a}^{-1} \iota_{\sharp_a(\mathrm{d}\beta)} \mathrm{d}\gamma, \\ &= (-1)^{\deg_H \beta} \frac{(b-a)!(c+a-(n+1))!}{(b+c-(n+1))!} \iota_{\sharp_a(\mathrm{d}\beta)} \mathrm{d}\gamma \end{aligned}$$

for every $0 \leq a \leq b$, and $n+1-a \geq c$, since we have

$$\iota_{\sharp_a(\mathrm{d}\beta)} \mathrm{d}\gamma = \binom{b+c-(n+1)}{b-a} \iota_{\sharp_b(\mathrm{d}\beta)} \mathrm{d}\gamma.$$

Remark 3.1.3. Also note, as it follows from Lemma 3.1.1, that we have an explicit expression for $\sharp_a(\beta)$, namely

$$\sharp_a(\beta) = \sharp_b(\beta) \wedge \mathbb{1}_{b-a}.$$

This will result useful in the sequel, being the condition we will impose for compatibility among extensions.

3.2. *Second step*: *Extension of* $\sharp_1$ *to* $(S^1)^{\wedge a}$, *for* $a \geq 1$

We are now ready to start defining the extensions of the $\sharp$-mappings.

Theorem 3.2.1. *Let* $(M, S^a, \sharp_a)$ *be a regular graded Dirac manifold of order* n. *Then, there exists a unique extension of* $\sharp_1$,

$$(S^1)^{\wedge a} \xrightarrow{\widetilde{\sharp_1}} \bigwedge^{a-1} M \otimes_M \bigvee_n M / K_n$$

for all possible values of $a \geq 1$, such that for any $\alpha \in S^n$ and $\beta \in (S^1)^{\wedge a}$ we have

$$\iota_{\sharp_n(\alpha)}\beta = (-1)^{(n+1-a)}\iota_{\widetilde{\sharp}_1(\beta)}\alpha.$$

Proof. Note that the map $\alpha \otimes \beta \mapsto (-1)^{(n+1-a)}\iota_{\sharp_n(\alpha)}\beta$ for $\alpha \in S^n$, $\beta \in (S^1)^{\wedge a}$ defines a map $S^n \otimes (S^1)^{\wedge a} \to \bigwedge^{a-1} M$. Taking the adjoint (see Remark 3.1.1), we obtain $\widetilde{\sharp}_1 \colon (S^1)^{\wedge a} \xrightarrow{\widetilde{\sharp}_1} \bigwedge^{a-1} M \otimes \bigvee_n M/K_n$. By construction, it satisfies the required equality.

Furthermore, the equality $\iota_{\sharp_n(\alpha)}\beta = (-1)^{(n+1-a)}\iota_{\widetilde{\sharp}_1(\beta)}\alpha$ just means that $\widetilde{\sharp}_1$ is the adjoint (in the previous sense) of the first map defined. $\square$

Remark 3.2.1. Signs are chosen throughout the text in the $\sharp$ mappings so that the following equality holds: $\iota_{\sharp(\alpha)}\beta = (-1)^{(n+1-a)(n+1-b)}\iota_{\sharp(\beta)}\alpha$, maintaining skew-symmetry as in Definition 2.1.3. Note that this is the condition imposed in Theorem 3.2.1.

Remark 3.2.2. The previous extension of $\sharp_1$ recovers the construction by Michor in [41] and by Grabowski in [32]. Indeed, it is easily shown that the map from Proposition 3.1.1 satisfies

$$\widetilde{\sharp}_1(\theta_1 \wedge \cdots \wedge \theta_a) = (-1)^{a+1}\sum_{j=1}^{a}(-1)^{j+1}\theta_1 \wedge \cdots \wedge \hat{\theta}_j \wedge \cdots \wedge \theta_a \otimes \sharp_1(\theta_j),$$

so that it defines an anti-derivation, which is the main property used in both cited papers for their respective construction. We prefer the point of view of the proof of Proposition 3.1.1 because of its immediate generalization to different degrees, which is not so easily accomplished from the point of view of derivations.

As a corollary of Proposition 3.1.1, we can define an extension of brackets for Hamiltonian $(n-1)$-forms and forms of arbitrary degree in a suitable subspace: Let

$$\Omega^{a-1}_H(M)[1] := \{\alpha \in \Omega^{a-1}(M) : \mathrm{d}\alpha \in (S^1)^{\wedge a}\}.$$

The extension of $\sharp_1$ defines a bracket

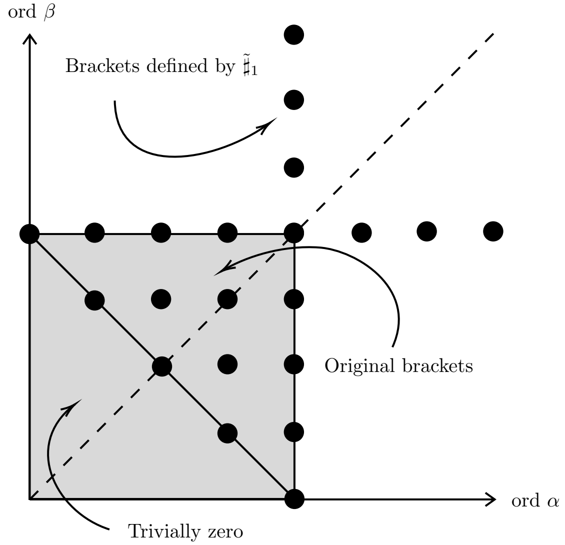
$$\Omega^{n-1}_H(M) \otimes \Omega^{a-1}_H(M)[1] \xrightarrow{\{\cdot,\cdot\}} \Omega^{a-1}(M)$$

by putting

$$\{\alpha, \Theta\} := (-1)^{\deg_H \Theta}\iota_{\widetilde{\sharp}_1(\mathrm{d}\Theta)}\mathrm{d}\alpha. \tag{3}$$

By Proposition 3.1.1, this bracket coincides with the original one for Hamiltonian forms $\Theta \in \Omega^{b-1}_H(M)$, $1 \leq b \leq n$.

Remark 3.2.3. The diagram below shows the achieved extension with this first step: There is a point on the coordinate plane $(\deg\alpha, \deg\beta)$ if we can evaluate a bracket on a suitable pair of forms of such an order. In the gray square, we have represented the original bracket of Hamiltonian forms, and we have extended it to the case $(n-1, a)$ or $(a, n-1)$, for arbitrary $a \geq 0$.



We now turn into studying elementary properties of the previous extension, dealing with the problem of how many properties of Definition 2.1.2 remain.

We will find useful the fact that we may compute $\widetilde{\sharp}_1(\mathrm{d}\{\alpha, \Theta\})$, where $\alpha \in \Omega_H^{n-1}(M)$ and $\Theta \in \Omega_H^{a-1}(M)[1]$. Indeed, we have the following proposition.

Proposition 3.2.1. *Let* $\Theta \in \Omega_H^{a-1}(M)[1]$, *and* $\alpha \in \Omega_H^{n-1}(M)$. *Then,*

$$\widetilde{\sharp}_1(\mathrm{d}\{\alpha, \Theta\}) = -\pounds_{\sharp_n(\mathrm{d}\alpha)}(\widetilde{\sharp}_1(\mathrm{d}\Theta)).$$

Before proving the previous proposition, let us first show that the bracket is closed.

Lemma 3.2.1. *Given* $\alpha \in \Omega_H^{n-1}(M)$ *and* $\Theta \in \Omega^{a-1}(M)[1]$, *we have* $\{\alpha, \Theta\} \in \Omega^{a-1}(M)[1]$.

Proof. Let $\alpha \in \Omega_H^{n-1}(M)$ and $\Theta \in \Omega_H^{a-1}(M)[1]$. We need to check that $\{\alpha, \Theta\} \in \Omega_H^{a-1}(M)[1]$, which is equivalent to $\mathrm{d}\iota_{\sharp_n(\mathrm{d}\alpha)}\mathrm{d}\Theta \in (S^1)\wedge a$. Now, the last requirement is satisfied if and only if $\iota_{K_1}\mathrm{d}\iota_{\sharp_n(\mathrm{d}\alpha)}\mathrm{d}\Theta = 0$, where $K_1 = (S^1)^{\circ,1}$. Indeed, let X be a vector field taking values in K_1. Then,

$$\begin{aligned}
\iota_X \mathrm{d}\iota_{\sharp_n(\mathrm{d}\alpha)}\mathrm{d}\Theta &= \pounds_X \mathrm{d}\iota_{\sharp_n(\mathrm{d}\alpha)}\mathrm{d}\Theta - \mathrm{d}\iota_X\iota_{\sharp_n(\mathrm{d}\alpha)}\mathrm{d}\Theta = \pounds_X \mathrm{d}\iota_{\sharp_n(\mathrm{d}\alpha)}\mathrm{d}\Theta \\
&= \mathrm{d}\pounds_X\iota_{\sharp_n(\mathrm{d}\alpha)}\mathrm{d}\Theta = \mathrm{d}\iota_{[X,\sharp_n(\mathrm{d}\alpha)]}\mathrm{d}\Theta + \mathrm{d}\iota_{\sharp_n(\mathrm{d}\alpha)}\pounds_X\mathrm{d}\Theta \\
&= \mathrm{d}\iota_{[X,\sharp_n(\mathrm{d}\alpha)]}\mathrm{d}\Theta,
\end{aligned}$$

which is zero, as $[X, \sharp_n(\mathrm{d}\alpha)]$ takes values in K_1, since the graded Dirac structure is integrable. $\square$

Proof of Proposition 3.2.1. In order to prove the equality, let us check that contraction against an arbitrary form β taking values in S^n by both sides of the desired equality yields the same result. Indeed, using the defining property of $\widetilde{\sharp}_1$ (see Theorem 3.2.1) and that $\mathrm{d}\Theta \in (S^1)^{\wedge a}$, we get

$$\begin{aligned}\iota_{\widetilde{\sharp}_1(\mathrm{d}\{\alpha,\Theta\})}\beta &= (-1)^{n+1-a}\iota_{\sharp_n(\beta)}\mathrm{d}\{\alpha,\Theta\} = -\iota_{\sharp_n(\beta)}\mathrm{d}\iota_{\widetilde{\sharp}_1(\mathrm{d}\Theta)}\mathrm{d}\alpha\\ &= -\pounds_{\sharp_n(\beta)}\iota_{\widetilde{\sharp}_1(\mathrm{d}\Theta)}\mathrm{d}\alpha + \mathrm{d}\iota_{\sharp_n(\beta)}\iota_{\widetilde{\sharp}_1(\mathrm{d}\Theta)}\mathrm{d}\alpha\\ &= -(-1)^{(n+1-a)}\pounds_{\sharp_n(\beta)}\iota_{\sharp_n(\mathrm{d}\alpha)}\mathrm{d}\Theta + \mathrm{d}\iota_{\sharp_n(\beta)}\iota_{\widetilde{\sharp}_1(\mathrm{d}\Theta)}\mathrm{d}\alpha.\end{aligned}$$

Now, rewriting the leftmost term employing the formula $\pounds_X\iota_Y\omega = \iota_{[X,Y]}\omega + \iota_Y\pounds_X\omega$, we get

$$\begin{aligned}\iota_{\widetilde{\sharp}_1(\mathrm{d}\{\alpha,\Theta\})}\beta = -(-1)^{n+1-a}\big(&\iota_{[\sharp_n(\beta),\sharp_n(\mathrm{d}\alpha)]}\mathrm{d}\Theta\\ &+ \iota_{\sharp_n(\mathrm{d}\alpha)}\pounds_{\sharp_n(\beta)}\mathrm{d}\Theta\big) + \mathrm{d}\iota_{\sharp_n(\beta)}\iota_{\widetilde{\sharp}_1(\mathrm{d}\Theta)}\mathrm{d}\alpha.\end{aligned} \tag{4}$$

Recall that $(S^a, \sharp_a)$ is a graded Dirac structure, so that we have $[\sharp_n(\beta), \sharp_n(\mathrm{d}\alpha)] = \sharp_n(\eta)$, where

$$\eta = \pounds_{\sharp_n(\beta)}\mathrm{d}\alpha - \pounds_{\sharp_n(\mathrm{d}\alpha)}\beta + \frac{1}{2}\,\mathrm{d}\big(\iota_{\sharp_n(\mathrm{d}\alpha)}\beta - \iota_{\sharp_n(\beta)}\mathrm{d}\alpha\big) = -\pounds_{\sharp_n(\mathrm{d}\alpha)}\beta. \tag{5}$$

Using Eq. (4) together with Eq. (5) we obtain

$$\begin{aligned}\iota_{\widetilde{\sharp}_1(\mathrm{d}\{\alpha,\Theta\})}\beta &= -(-1)^{n+1-a}\big(\iota_{[\sharp_n(\beta),\sharp_n(\mathrm{d}\alpha)]}\mathrm{d}\Theta + \iota_{\sharp_n(\mathrm{d}\alpha)}\pounds_{\sharp_n(\beta)}\mathrm{d}\Theta\big) + \mathrm{d}\iota_{\sharp_n(\beta)}\iota_{\widetilde{\sharp}_1(\mathrm{d}\Theta)}\mathrm{d}\alpha\\ &= -(-1)^{n+1-a}\big(\iota_{\sharp_n(\eta)}\mathrm{d}\Theta + \iota_{\sharp_n(\mathrm{d}\alpha)}\pounds_{\sharp_n(\beta)}\mathrm{d}\Theta\big) + \mathrm{d}\iota_{\sharp_n(\beta)}\iota_{\widetilde{\sharp}_1(\mathrm{d}\Theta)}\mathrm{d}\alpha\\ &= -(-1)^{n+1-a}\big((-1)^{n+1-a}\iota_{\widetilde{\sharp}_1(\mathrm{d}\Theta)}\eta + \iota_{\sharp_n(\mathrm{d}\alpha)}\pounds_{\sharp_n(\beta)}\mathrm{d}\Theta\big)\\ &\quad + \mathrm{d}\iota_{\sharp_n(\beta)}\iota_{\widetilde{\sharp}_1(\mathrm{d}\Theta)}\mathrm{d}\alpha\\ &= \iota_{\widetilde{\sharp}_1(\mathrm{d}\Theta)}\pounds_{\sharp_n(\mathrm{d}\alpha)}\beta - (-1)^{n+1-a}\iota_{\sharp_n(\mathrm{d}\alpha)}\pounds_{\sharp_n(\beta)}\mathrm{d}\Theta + \mathrm{d}\iota_{\sharp_n(\beta)}\iota_{\widetilde{\sharp}_1(\mathrm{d}\Theta)}\mathrm{d}\alpha.\end{aligned}$$

Now it only remains to rewrite $\iota_{\widetilde{\sharp}_1(\mathrm{d}\alpha)}\pounds_{\sharp_n(\mathrm{d}\alpha)}\beta = -\iota_{\pounds_{\sharp_n(\mathrm{d}\alpha)}\widetilde{\sharp}_1(\mathrm{d}\Theta)}\beta + \pounds_{\sharp_n(\mathrm{d}\alpha)}\iota_{\widetilde{\sharp}_1(\mathrm{d}\Theta)}\beta$ to get

$$\begin{aligned}\iota_{\widetilde{\sharp}_1(\mathrm{d}\{\alpha,\Theta\})}\beta &= -\iota_{\pounds_{\sharp_n(\mathrm{d}\alpha)}\widetilde{\sharp}_1(\mathrm{d}\Theta)}\beta + \pounds_{\sharp_n(\mathrm{d}\alpha)}\iota_{\widetilde{\sharp}_1(\mathrm{d}\Theta)}\beta\\ &\quad - (-1)^{n+1-a}\iota_{\sharp_n(\mathrm{d}\alpha)}\pounds_{\sharp_n(\beta)}\mathrm{d}\Theta + \mathrm{d}\iota_{\sharp_n(\beta)}\iota_{\widetilde{\sharp}_1(\mathrm{d}\Theta)}\mathrm{d}\alpha\\ &= -\iota_{\pounds_{\sharp_n(\mathrm{d}\alpha)}\widetilde{\sharp}_1(\mathrm{d}\Theta)}\beta,\end{aligned}$$

finishing the proof. ∎

Now we turn into studying its properties under contraction by vector fields.

Proposition 3.2.2. *Let α be an a-form taking values in $(S^1)^{\wedge a}$ and let $X \in \mathfrak{X}(M)$. Then, $\iota_X \alpha \in (S^1)^{\wedge(a-1)}$ and $\widetilde{\sharp}_1(\iota_X\alpha) = -\iota_X \widetilde{\sharp}_1(\alpha)$, where contraction is interpreted as $\iota_X(\theta \otimes u) = (\iota_X\theta) \otimes u$.*

Proof. Indeed, let $\alpha = \theta_1 \wedge \cdots \wedge \theta_a$, where $\theta_i \in S^1$ for every $i = 1, \dots, a$. Then $\widetilde{\sharp}_1(\theta_1 \wedge \cdots \wedge \theta_a) = (-1)^{a+1} \sum_{j=1}^{a} (-1)^{j+1} \theta_1 \wedge \cdots \wedge \hat{\theta}_j \wedge \cdots \wedge \theta_a \otimes \sharp_1(\theta_j)$, so that

$$
\begin{aligned}
&\iota_X \widetilde{\sharp}_1(\theta_1 \wedge \cdots \wedge \theta_a) \\
&\quad = (-1)^{a+1} \sum_{j=1}^{a} \sum_{i<j} \theta_i(X) \theta_1 \wedge \cdots \wedge \hat{\theta}_i \wedge \cdots \wedge \hat{\theta}_j \wedge \cdots \wedge \theta_a \otimes \sharp_1(\theta_j) \\
&\qquad + (-1)^{a} \sum_{j=1}^{a} \sum_{i>j} \theta_i(X) \theta_1 \wedge \cdots \wedge \hat{\theta}_j \wedge \cdots \wedge \hat{\theta}_i \wedge \cdots \wedge \theta_a \otimes \sharp_1(\theta_j).
\end{aligned}
$$

Now, $\iota_X(\theta_1 \wedge \cdots \wedge \theta_a) = \sum_{i=1}^{a} (-1)^{j+1} \theta_i(X) \theta_1 \wedge \cdots \wedge \hat{\theta}_i \wedge \cdots \wedge \theta_a$ and

$$
\begin{aligned}
&\widetilde{\sharp}_1(\iota_X(\theta_1 \wedge \cdots \wedge \theta_a)) \\
&\quad = (-1)^{a} \sum_{i=1}^{a} \sum_{j>i} \theta_i(X) \theta_1 \wedge \cdots \wedge \hat{\theta}_i \wedge \cdots \wedge \hat{\theta}_j \wedge \cdots \wedge \theta_a \otimes \sharp_1(\theta_j) \\
&\qquad + (-1)^{a+1} \sum_{i=1}^{a} \sum_{j<i} \theta_i(X) \theta_1 \wedge \cdots \wedge \hat{\theta}_j \wedge \cdots \wedge \hat{\theta}_i \wedge \cdots \wedge \theta_a \otimes \sharp_1(\theta_j),
\end{aligned}
$$

which finishes the proof. $\square$

Theorem 3.2.2. *The extension of the graded Poisson bracket on a graded Dirac manifold defined by the extension $\widetilde{\sharp}_1$ satisfies the following properties:*

(i) *It is graded-skew-symmetric.*
(ii) *It satisfies invariance by symmetries: If $\pounds_X\Theta = 0$, for some vector field $X \in \mathfrak{X}(M)$ and differential form $\Theta \in \Omega_H^{a-1}(M)[1]$, then $\{\alpha, \iota_X\Theta\} = -\iota_X\{\alpha, \Theta\}$.*
(iii) *It satisfies the (graded) Jacobi identity for $\alpha, \beta \in \Omega_H^{n-1}(M)$, and $\Theta \in \Omega_H^{a-1}(M)[1]$ up to an exact term:*

$$\{\alpha, \{\beta, \Theta\}\} + \{\Theta, \{\alpha, \beta\}\} + \{\beta, \{\Theta, \alpha\}\} = \textit{exact form.}$$

(iv) *It satisfies the Leibniz identity, for $\alpha \in \Omega_H^{n-1}(M)$, $\Theta_1 \in \Omega_H^{a_1-1}(M)[1]$ and $\Theta_2 \in \Omega_H^{a_2-1}(M)[1]$, we have*

$$\{\alpha, \Theta_1 \wedge \mathrm{d}\Theta_2\} = \{\alpha, \Theta_1\} \wedge \mathrm{d}\Theta_2 + (-1)^{a_1} \mathrm{d}\Theta_1 \wedge \{\alpha, \Theta_2\}.$$

Proof. (i) It is satisfied by definition.

(ii) Indeed, suppose $\pounds_X\Theta = \mathrm{d}\iota_X\Theta + \iota_X \mathrm{d}\Theta = 0$. Then, for every $\alpha \in \Omega^{n-1}_H(M)$, we have

$$\begin{aligned}\{\alpha, \iota_X\Theta\} &= (-1)^{\deg_H \Theta + 1}\iota_{\widetilde{\sharp}_1(\mathrm{d}\iota_X\Theta)}\mathrm{d}\alpha = (-1)^{\deg_H \Theta}\iota_{\widetilde{\sharp}_1(\iota_X \mathrm{d}\Theta)}\mathrm{d}\alpha \\ &= -(-1)^{\deg_H \Theta}\iota_{\iota_X\widetilde{\sharp}_1(\mathrm{d}\Theta)}\mathrm{d}\alpha = -(-1)^{\deg_H \Theta}\iota_X\iota_{\widetilde{\sharp}_1(\mathrm{d}\Theta)}\mathrm{d}\alpha \\ &= -\iota_X\{\alpha, \Theta\},\end{aligned}$$

where in the third equality we have used Proposition 3.2.2.

(iii) Finally, for $\alpha, \beta \in \Omega^{n-1}_H(M)$ and $\Theta \in \Omega^{a-1}_H(M)[1]$, we have

$$\begin{aligned}\{\{\alpha,\beta\},\Theta\} &= (-1)^{n-a}\iota_{\widetilde{\sharp}_1(\mathrm{d}\Theta)}\mathrm{d}\{\alpha,\beta\} = -\iota_{\sharp_n(\mathrm{d}\{\alpha,\beta\})}\mathrm{d}\Theta = \iota_{[\sharp_n(\mathrm{d}\alpha),\sharp_n(\mathrm{d}\beta)]}\mathrm{d}\Theta \\ &= \pounds_{\sharp_n(\mathrm{d}\alpha)}\iota_{\sharp_n(\mathrm{d}\beta)}\mathrm{d}\Theta - \iota_{\sharp_n(\mathrm{d}\beta)}\pounds_{\sharp_n(\mathrm{d}\alpha)}\mathrm{d}\Theta \\ &= \mathrm{d}\iota_{\sharp_n(\mathrm{d}\alpha)}\iota_{\sharp_n(\mathrm{d}\beta)}\mathrm{d}\Theta + \iota_{\sharp_n(\mathrm{d}\alpha)}\mathrm{d}\iota_{\sharp_n(\mathrm{d}\beta)}\mathrm{d}\Theta - \iota_{\sharp_n(\mathrm{d}\beta)}\mathrm{d}\iota_{\sharp_n(\mathrm{d}\alpha)}\mathrm{d}\Theta \\ &= \mathrm{d}\iota_{\sharp_n(\mathrm{d}\alpha)}\iota_{\sharp_n(\mathrm{d}\beta)}\mathrm{d}\Theta + \{\{\Theta,\beta\},\alpha\} - \{\{\Theta,\alpha\},\beta\},\end{aligned}$$

frow which Jacobi identity follows.

(iv) First notice, as $\mathrm{d}\Theta_i \in (S^1)^{\wedge a_i}$ that we have $\mathrm{d}\Theta_1 \wedge \mathrm{d}\Theta_2 \in (S^1)^{\wedge(a_1+a_2)}$. Now, we have

$$\begin{aligned}\{\alpha, \Theta_1 \wedge \mathrm{d}\Theta_2\} &= (-1)^{\deg_H(\Theta_1\wedge \mathrm{d}\Theta_2)}\iota_{\widetilde{\sharp}_1(\mathrm{d}\Theta_1\wedge\mathrm{d}\Theta_2)}\mathrm{d}\alpha = -\iota_{\sharp_n(\mathrm{d}\alpha)}(\mathrm{d}\Theta_1 \wedge \mathrm{d}\Theta_2) \\ &= -\{\Theta_1,\alpha\} \wedge \mathrm{d}\Theta_2 + (-1)^{a_1+1}\mathrm{d}\Theta_1 \wedge \{\Theta_2,\alpha\} \\ &= \{\alpha,\Theta_1\} \wedge \mathrm{d}\Theta_2 + (-1)^{a_1}\mathrm{d}\Theta_1 \wedge \{\alpha,\Theta_2\}.\end{aligned}$$ □

Remark 3.2.4. Note that we cannot use the values of $\widetilde{\sharp}_1$ to define a bracket on

$$\Omega^{a-1}_H(M) \otimes \Omega^{b-1}_H M[1] \xrightarrow{\{\cdot,\cdot\}} \Omega^{a+b-(n+1)}(M)$$

for $a < n$, as it would be trivially zero.

3.3. *Last step*: *Extension of* $\sharp_a$ *to a suitable family of forms*

For the sake of exposition, let us first discuss how to extend the brackets of the previous step to a bracket $\{\alpha, \Theta\}$, where $\alpha \in \Omega^{n-2}_H(M)$ and $\Theta \in \Omega^{a-1}_H(M)[1]$. Following the proposed approach, we would need to find an extension

$$\widetilde{\sharp}_2 : (S^1)^{\wedge a} \to \bigwedge^{a-2} M \otimes \bigvee_{n-1} M/K_{n-1}.$$

In order to define the new extension by Eq. (3), we need to ask that for compatibility with $\widetilde{\sharp}_1$, so that we require $\widetilde{\sharp}_1 = \widetilde{\sharp}_2 \wedge \mathbb{1}_1$ (following the same idea behind the formula in Remark 3.1.3), where the exterior product is understood component wise. An analogous computation to that in Lemma 3.1.1 shows that the previous requirement is equivalent to $\widetilde{\sharp}_2$ satisfying

$$\iota_{\widetilde{\sharp}_2(\Theta)}\alpha = \iota_{\widetilde{\sharp}_1(\Theta)}\alpha.$$

It could happen that such extension does not exist and thus, we need to restrict to the following subspace, which we shall assume to be a subbundle:

$$S^a[2] = \left\{ \Theta \in (S^1)^{\wedge a} : \exists \widetilde{\sharp}_2(\Theta) \in \bigwedge^{a-2} M \right.$$
$$\left. \otimes \bigwedge_{n-1} M/K_{n-1} \text{ with } \iota_{\widetilde{\sharp}_2(\Theta)}\alpha = \iota_{\widetilde{\sharp}_1(\Theta)}\alpha,\, \forall\, \alpha \in S^n \right\}.$$

Following the notation of the previous section, we obtain a bracket

$$\Omega_H^{n-2}(M) \otimes \Omega_H^{a-1}(M)[2] \otimes \xrightarrow{\{\cdot,\cdot\}_{\widetilde{\sharp}_2}} \Omega^{a-2}(M),$$

where

$$\Omega_H^{a-1}(M)[2] := \{\alpha \in \Omega^{a-1}(M) : \mathrm{d}\alpha \in S^a[2]\}, \quad \{\alpha, \Theta\}_{\widetilde{\sharp}_2} = (-1)^{\deg_H \Theta}\iota_{\widetilde{\sharp}_2(\Theta)}\mathrm{d}\alpha.$$

We now proceed iteratively and define

$$S^a[j] := \left\{ \Theta \in (S^1)^{\wedge a} : \exists \widetilde{\sharp}_j(\Theta) \in \bigwedge^{a-j} M \right.$$
$$\left. \otimes \bigwedge_{n+1-j} M/K_{n+1-j} \text{ with } \iota_{\widetilde{\sharp}_j(\Theta)}\alpha = \iota_{\widetilde{\sharp}_1(\Theta)}\alpha,\, \forall\, \alpha \in S^n \right\} \tag{6}$$

and then choose a map

$$\widetilde{\sharp}_j : S^a[j] \to \bigwedge^{a-j} M \otimes \bigvee_{n+1-j} M/K_{n+1-j},$$

that is compatible with all the previous choices $\widetilde{\sharp}_1, \widetilde{\sharp}_2, \dots, \widetilde{\sharp}_{j-1}$ (compatible meaning $\widetilde{\sharp}_i = C(i,j)\widetilde{\sharp}_j \wedge \mathbb{1}_{j-i}$, for a suitable constant $C(i,j)$ to be specified later) to then extend the bracket to

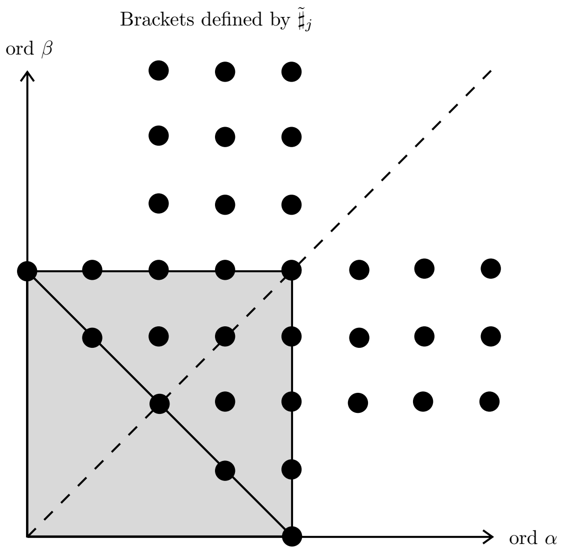
$$\Omega_H^{n-j}(M) \otimes \Omega_H^{a-1}(M)[j] \xrightarrow{\{\cdot,\cdot\}} \Omega^{a-j}(M), \quad \{\alpha, \Theta\}_{\widetilde{\sharp}_j} := (-1)^{\deg_H \Theta}\iota_{\widetilde{\sharp}_j(\mathrm{d}\Theta)}\mathrm{d}\alpha, \tag{7}$$

where

$$\Omega_H^{a-1}(M)[j] = \{\Theta \in \Omega^{a-1}(M) : \mathrm{d}\Theta \in S^a[j]\} \tag{8}$$

with $a \geq j$. The following diagram represents the pairs of forms on which we can evaluate the bracket given the extension $\widetilde{\sharp}_j$, extending by skew-symmetry and using

Eq. (7):



The previous discussion motivates the following definition.

Definition 3.3.1 (Extension). A collection of maps

$$\widetilde{\sharp}_i \colon S^a[i] \to \bigwedge^{a-i} M \otimes \bigvee_{n+1-i} M/K_{n+1-i}$$

for $a \geq i$ is called an extension of $\sharp_i$ if we have $\widetilde{\sharp}_1 = \widetilde{\sharp}_i \wedge \mathbb{1}_{i-1}$ (so that the compatibility condition is satisfied).

Remark 3.3.1. The reader may notice that the definition of the bundle $S^a[j]$ does not depend on the mappings $\widetilde{\sharp}_2, \dots, \widetilde{\sharp}_{j-1}$, and may correctly point out that an extension of $\widetilde{\sharp}_j$ defined in the whole bundle $S^a[j]$ that is compatible with the previous extensions may not exist. This would mean that an extension of the brackets (following our construction) compatible with the previous extensions is not possible.

Let us give some initial results concerning these vector subbundles.

Lemma 3.3.1. *Let V be a finite dimensional vector space and let $\mathbb{1}_a$ denote the identity on $\bigwedge^a V^* \otimes \bigwedge^a V$. Then,*

$$\mathbb{1}_a \wedge \mathbb{1}_b = \binom{a+b}{a} \mathbb{1}_{a+b}.$$

Proof. Let $\{e_i\}$ denote a basis of V so that

$$\mathbb{1}_a = e^{i_1,\dots,i_a} \otimes e_{i_1,\dots,i_a}, \quad \mathbb{1}_b = e^{i_1,\dots,i_b} \otimes e_{i_1,\dots,i_b}.$$

Hence,

$$\mathbb{1}_a \wedge \mathbb{1}_b = e^{i_1,\ldots,i_a,j_1,\ldots,j_b} \otimes e_{i_1,\ldots,i_a,j_1,\ldots,j_b} = \delta^{\widetilde{I}\widetilde{J}}_{\widetilde{\mu}} \delta^{\widetilde{\nu}}_{\widetilde{I}\widetilde{J}} e^{\widetilde{\mu}} \otimes e_{\widetilde{\nu}} = \binom{a+b}{a} \mathbb{1}_{a+b},$$

which ends the proof. □

Proposition 3.3.1. *For $(M, S^a, \sharp_a)$ a regular graded Dirac manifold we have the following*:

(i) *Let $\widetilde{\sharp}_j : S^a[j] \to \bigwedge^{a-j} M \otimes \bigvee_{n+1-j} M/K_{n+1-j}$ be a vector bundle map satisfying $\iota_{\widetilde{\sharp}_j(\alpha)}\beta = \iota_{\widetilde{\sharp}_1(\alpha)}\beta$ for every $\beta \in S^n$. Then, there exists a unique extension $\widetilde{\sharp}_i : S^a[j] \to \bigwedge^{a-i} M \otimes \bigvee_{n+1-i} M/K_{n+1-i}$, for $1 \leq i < j$, satisfying the following two properties*:

(a) *For every $\gamma \in S^{n+1-i}$ and $\alpha \in S^a[j]$, we have*

$$\iota_{\widetilde{\sharp}_i(\alpha)}\gamma = \binom{j-1}{j-i} \iota_{\widetilde{\sharp}_j(\alpha)}\gamma;$$

(b) *For every $\beta \in S^n$, and $\alpha \in S^a[j]$, we must have $\iota_{\widetilde{\sharp}_i(\alpha)}\beta = \iota_{\widetilde{\sharp}_1(\alpha)}\beta$.*

(ii) *$S^a[j+1] \subseteq S^a[j]$, for $1 \leq j < n$ and $1 \leq a$.*

(iii) *For $X \in \mathfrak{X}(M)$, we have $\iota_X : S^a[j] \to S^{a-1}[j-1]$ for every $a > n$ and $2 \leq j \leq n$.*

Proof. (i) Let $\alpha \in S^a[j]$. Then, we can define a map $S^{n+1-i} \to \bigwedge^{a-i} M$ by $\gamma \mapsto \binom{j-1}{j-i}^{-1} \iota_{\widetilde{\sharp}_j(\alpha)}\gamma$. Taking the adjoint of this map, we obtain an element $\widetilde{\sharp}_i(\alpha) \in \bigwedge^{a-i} M \otimes \bigvee_{n+1-i} M/K_{n+1-i}$. This clearly satisfies the first property. In order to check the second, let us recall that the compatibility conditions read as

$$\widetilde{\sharp}_1 = \widetilde{\sharp}_i \wedge \mathbb{1}_{i-1},$$

which holds in this case since we have $\widetilde{\sharp}_i = \binom{j-1}{j-i}^{-1} \widetilde{\sharp}_j \wedge \mathbb{1}_{j-i}$ (by Lemma 3.3.1), $\mathbb{1}_{j-i} \wedge \mathbb{1}_{i-1} = \binom{j-1}{j-i} \mathbb{1}_{j-1}$, and $\widetilde{\sharp}_1 = \widetilde{\sharp}_j \wedge \mathbb{1}_{j-1}$.

(ii) The construction applied in item (i) clearly proves the inclusion.

(iii) Let $\alpha \in S^a[j]$, so that there exists $\widetilde{\sharp}_j(\alpha) \in \bigwedge^{a-j} M \otimes \bigvee_{n+1-j} M/K_{n+1-j}$ such that $\iota_{\widetilde{\sharp}_j(\alpha)}\beta = \iota_{\widetilde{\sharp}_1(\alpha)}\beta$ for every $\beta \in S^n$. Define $\widetilde{\sharp}_{j-1}(\iota_X\alpha) := -\iota_X \widetilde{\sharp}_{j-1}(\alpha) - (-1)^{a-j} \widetilde{\sharp}_j(\alpha) \wedge X$, where $\widetilde{\sharp}_{j-1}(\alpha)$ denotes the unique extension of $\widetilde{\sharp}_j(\alpha)$ from item (i). Then, for arbitrary $\beta \in S^n$ we have

$$\begin{aligned}
\iota_{\widetilde{\sharp}_j(\iota_x\alpha)}\beta &= -\iota_{\iota_X \widetilde{\sharp}_j(\alpha)}\beta - (-1)^{a-j} \iota_{\widetilde{\sharp}_j(\alpha)\wedge X}\beta = -\iota_{\iota_X \widetilde{\sharp}_j(\alpha)}\beta - (-1)^{n+1+a} \iota_{\widetilde{\sharp}_j(\alpha)} \iota_X \beta \\
&= -\iota_X(\iota_{\widetilde{\sharp}_j(\alpha)}\beta) + (-1)^{n+1+a} \iota_{\widetilde{\sharp}_j(\alpha)} \iota_X \beta - (-1)^{n+1+a} \iota_{\widetilde{\sharp}_j(\alpha)} \iota_X \beta \\
&= -\iota_X(\iota_{\widetilde{\sharp}_1(\alpha)}\beta) = -\iota_{\iota_X \widetilde{\sharp}_1(\alpha)}\beta \\
&= \iota_{\widetilde{\sharp}_1(\iota_X\alpha)}\beta,
\end{aligned}$$

where in the last equality we have used Proposition 3.2.2. □

Let $m = \dim M$. Then, Proposition 3.3.1 gives the following picture of the subbundles $S^a[j]$ and the relations between them:

$$
\begin{tikzcd}
S^m[n] \arrow[r, hook] & S^m[n-1] \arrow[r, hook] & \cdots \arrow[r, hook] & S^m[j] \arrow[r, hook] & S^m[j-1] \arrow[r, hook] & \cdots \arrow[r, hook] & S^m[1] \\
\vdots \arrow[dr, "\iota_{\mathrm{T}M}"] & \vdots & \ddots \arrow[dr, "\iota_{\mathrm{T}M}"] & \vdots \arrow[dr, "\iota_{\mathrm{T}M}"] & \vdots & \ddots \arrow[dr, "\iota_{\mathrm{T}M}"] & \vdots \\
S^a[n] \arrow[r, hook] \arrow[dr, "\iota_{\mathrm{T}M}"] & S^a[n-1] \arrow[r, hook] & \cdots \arrow[r, hook] \arrow[dr, "\iota_{\mathrm{T}M}"] & S^a[j] \arrow[r, hook] \arrow[dr, "\iota_{\mathrm{T}M}"] & S^a[j-1] \arrow[r, hook] & \cdots \arrow[r, hook] \arrow[dr, "\iota_{\mathrm{T}M}"] & S^a[1] \\
S^{a-1}[n] \arrow[r, hook] & S^{a-1}[n-1] \arrow[r, hook] & \cdots \arrow[r, hook] & S^{a-1}[j] \arrow[r, hook] & S^{a-1}[j-1] \arrow[r, hook] & \cdots \arrow[r, hook] & S^{a-1}[1] \\
\vdots \arrow[dr, "\iota_{\mathrm{T}M}"] & \vdots & \ddots \arrow[dr, "\iota_{\mathrm{T}M}"] & \vdots \arrow[dr, "\iota_{\mathrm{T}M}"] & \vdots & \ddots \arrow[dr, "\iota_{\mathrm{T}M}"] & \vdots \\
S^{n+1}[n] \arrow[r, hook] & S^{n+1}[n-1] \arrow[r, hook] & \cdots \arrow[r, hook] & S^{n+1}[j] \arrow[r, hook] & S^{n+1}[j-1] \arrow[rr, hook] & & S^{n+1}[1]
\end{tikzcd}
.
$$

We have horizontal arrows representing inclusions from item (ii), and diagonal arrows representing contraction by vector fields from item (iii).

Therefore, by Proposition 3.3.1, we can only hope of finding a global extension if we define a final vector bundle map $\widetilde{\sharp}_n : S^a[n] \to \bigwedge^{a-n} M \otimes \mathrm{T}M/K_1$, for $a \geq n$. Then, we may obtain the associated extensions as

$$
\widetilde{\sharp}_i = \binom{n-1}{n-i}^{-1} \widetilde{\sharp}_n \wedge \mathbb{1}_{n-i} \tag{9}
$$

and defining the space of Hamiltonian forms as

$$
\Omega^{a-1}_H(M)[n] := \begin{cases} \Omega^{a-1}_H(M) & a \leq n-1, \\ \{\alpha \in \Omega^a(M) \colon \mathrm{d}\alpha \in S^a[n]\} & a \geq n, \end{cases}
$$

we may define a bracket

$$
\Omega^{a-1}(M)[n] \otimes \Omega^{b-1}(M)[n] \xrightarrow{\{\cdot,\cdot\}_{\widetilde{\sharp}_n}} \Omega^{a+b-(n+1)}(M),
$$

as follows:

$$
\{\Theta_1, \Theta_2\}_{\widetilde{\sharp}_n} := \begin{cases} \{\Theta_1, \Theta_2\} & \text{if } \deg \Theta_1, \deg \Theta_2 \leq n-1, \\ (-1)^{\deg_H \Theta_2} \iota_{\widetilde{\sharp}_n(\mathrm{d}\Theta_2)} \mathrm{d}\Theta_1 & \text{if } \deg \Theta_2 \geq n. \end{cases} \tag{10}
$$

Remark 3.3.2. We will give dynamical interpretation of all possible extensions (compatible with a fibered structure and restricted to certain subfamily) in the following section (see Theorem 4.1.2).

Suppose that a final extension is chosen, $\widetilde{\sharp}_n \colon S^a[n] \to \bigwedge^{a-n} M \otimes \mathrm{T}M/K_1$. We now turn into the discussion of how many properties of Definition 2.1.2 remain valid for the bracket $\{\cdot,\cdot\}_{\widetilde{\sharp}_n}$, depending on the particular extension $\widetilde{\sharp}_n$ chosen. Unlike the bracket induced from $\widetilde{\sharp}_1$, these extensions do not satisfy all the properties of

Definition 2.1.2. Nevertheless, we can give sufficient conditions on the map $\widetilde{\sharp}_n$ so that these are satisfied:

Theorem 3.3.1. *Let $\widetilde{\sharp}_n : S^a[n] \to \bigwedge^{n-a} M \otimes \mathrm{T}M/K_1$ be an extension of $\sharp_n$ and denote by $\{\cdot,\cdot\}_{\widetilde{\sharp}_n}$ the induced bracket on $\Omega^a_H(M)[n]$. Then it satisfies the following properties*:

(i) *We have $\{\alpha,\Theta\}_{\widetilde{\sharp}_n} = -(-1)^{\deg_H \alpha \deg_H \Theta}\{\Theta,\alpha\}_{\widetilde{\sharp}_n}$ for $\alpha \in \Omega^a_H(M)$ usual Hamiltonian form and $\Theta \in \Omega^b_H(M)[n]$.*

(ii) *If $\widetilde{\sharp}_n$ satisfies $\widetilde{\sharp}_{n-1}(\iota_X\alpha) := -\iota_X\widetilde{\sharp}_{n-1}(\alpha) - (-1)^{a-n}\widetilde{\sharp}_n(\alpha)\wedge X$, where $\widetilde{\sharp}_{n-1}$ is the unique extension given by item (i) of Proposition 3.3.1, it satisfies invariance by symmetries: For $X \in \mathfrak{X}(M)$, if $\mathcal{L}_X\Theta = 0$, then $\{\Theta_1,\iota_X\Theta_2\} = -\iota_X\{\Theta_1,\Theta_2\}$ for every $\Theta_1 \in \Omega^a(M)[n]$, with $a \geq 2$.*

(iii) *If $\widetilde{\sharp}_n(\mathrm{d}\{\alpha,\Theta_1\}) = -\mathcal{L}_{\sharp_n(\mathrm{d}\alpha)}\widetilde{\sharp}_n(\mathrm{d}\Theta_1)$, then it satisfies a Jacobi identity*:

$$\{\alpha,\{\Theta_1,\Theta_2\}\} + \{\Theta_2,\{\alpha,\Theta_1\}\} + \{\Theta_1,\{\Theta_2,\alpha\}\} = \textit{exact form}$$

for $\alpha \in \Omega^{n-1}_H(M)$, $\Theta_1 \in \Omega^{a_1}(M)[n]$ and $\Theta_2 \in \Omega^{a_2}(M)[n]$.

Proof. (i) Graded skew-symmetry is given by definition.

(ii) Follows from a similar computation to that done in item (iii) of Proposition 3.3.1.

(iii) Again, follows from a similar computation from item (iii) of Theorem 3.2.2. □

4. Dynamics on Fibered Graded Dirac Manifolds

In this section, we apply the constructions of the previous section to define dynamics and study the evolution of arbitrary forms.

4.1. *Fibered extensions, Hamiltonians and Hamilton–De Donder–Weyl equations*

Let us first give a description of dynamics on graded Dirac manifolds, defining the equations of classical field theories in terms of Poisson brackets. We first need to define what we understand by a Hamiltonian, the object that will define the dynamics.

Let $\tau\colon M \to X$ be a fibered graded Dirac manifold (see Definition 2.3.2), whose $\sharp$ morphisms we denote by

$$\sharp_a\colon S^a \to \bigvee_{n+1-a} M/K_{n+1-a}.$$

We denote by $\widetilde{\sharp}_1$ the extension defined in Theorem 3.2.1 and by $\mathbb{1}_n$ the identity on $\bigwedge^n M \otimes \bigvee_n M$.

Definition 4.1.1 (Hamiltonian). A Hamiltonian is an n-form $\mathcal{H} \in \Omega^n_H(M)[n]$ such that $-\widetilde{\sharp}_1(\mathrm{d}\mathcal{H}) + \mathbb{1}_n$ is a semi-basic form that satisfies $\iota_{-\widetilde{\sharp}_1(\mathrm{d}\mathcal{H})+\mathbb{1}_n}\omega = \omega$ for every semi-basic n-form ω.

Remark 4.1.1. Recall that $\widetilde{\sharp}_1(\mathrm{d}\mathcal{H})$ is an n-multivector valued n-form, so that the expression in the above definition makes sense. Also, it still makes sense to talk about the notion of semi-basic (as well as that of r-horizontal) multi-vector valued forms.

Remark 4.1.2. The last condition $\iota_{-\widetilde{\sharp}_1(\mathrm{d}\mathcal{H})+\mathbb{1}_n}\omega = \omega$ is equivalent to $\iota_{\widetilde{\sharp}_1(\mathrm{d}\mathcal{H})}\omega = 0$ for every semi-basic n-form ω.

Remark 4.1.3. Note that S^n contains all semi-basic n-forms over X, so that K_n is contained in the space of 1-vertical multivectors, namely those multivectors taking values in $\langle \ker \mathrm{d}\tau \wedge \bigvee_{p-1} M\rangle$. Hence, it still makes sense to consider vertical valued sections of $\bigvee_n M/K_n$.

Example 4.1.1. Let $\pi\colon Y \to X$ be a fibered manifold. Let $\bigwedge^n_2 Y$ be endowed with its canonical multisymplectic structure and reduce it, following Example 2.3.1, to a fibered graded Dirac (in fact, Poisson) structure on $\tau\colon \bigwedge^n_2 Y / \bigwedge^n_1 Y \to X$. Then, given a section $h\colon \bigwedge^n_2 Y/\bigwedge^n_1 Y \to \bigwedge^n_2 Y$, we have that $\mathcal{H} := h^*\Theta$ is a Hamiltonian for the corresponding graded Poisson structure on $M = \bigwedge^n_2 Y/\bigwedge^n_1 Y$. Indeed, in canonical coordinates, we have $S^n = \langle \mathrm{d}^n x, \mathrm{d}y^i \wedge \mathrm{d}^{n-1}x_\mu, \mathrm{d}p^\mu_i \wedge \mathrm{d}^{n-1}x_\mu\rangle$ and

$$\sharp_n(\mathrm{d}^n x) = 0, \quad \sharp_n(\mathrm{d}y^i \wedge \mathrm{d}^{n-1}x_\mu) = -\frac{\partial}{\partial p^\mu_i}, \quad \sharp_n(\mathrm{d}p^\mu_i \wedge \mathrm{d}^{n-1}x_\mu) = \frac{\partial}{\partial y^i}.$$

An immediate computation shows $\mathcal{H} = H\mathrm{d}^n x - p^\mu_i \mathrm{d}y^i \wedge \mathrm{d}^{n-1}x_\mu$ so that

$$\mathrm{d}\mathcal{H} = \frac{\partial H}{\partial y^i}\mathrm{d}y^i \wedge \mathrm{d}^n x + \frac{\partial H}{\partial p^\mu_i}\mathrm{d}p^\mu_i \wedge \mathrm{d}^n x - \mathrm{d}p^\mu_i \wedge \mathrm{d}y^i \wedge \mathrm{d}^{n-1}x_\mu.$$

We have (for complete details regarding these computations, we refer to Sec. 5)

$$\begin{aligned}\widetilde{\sharp}_1(\mathrm{d}\mathcal{H}) = {} & \frac{1}{n}\frac{\partial H}{\partial y^i}\mathrm{d}^n x \otimes \frac{\partial}{\partial p^\mu_i} \wedge \frac{\partial^{n-1}}{\partial^{n-1}x_\mu} - \frac{\partial H}{\partial p^\mu_i}\mathrm{d}^n x \otimes \frac{\partial}{\partial y^i} \wedge \frac{\partial^{n-1}}{\partial^{n-1}x_\mu} \\ & + \mathrm{d}y^i \wedge \mathrm{d}^{n-1}x_\mu \otimes \frac{\partial}{\partial y^i} \wedge \frac{\partial^{n-1}}{\partial^{n-1}x_\mu} + \frac{1}{n}\mathrm{d}p^\mu_i \wedge \mathrm{d}^{n-1}x_\mu \otimes \frac{\partial}{\partial p^\mu_i} \wedge \frac{\partial^{n-1}}{\partial^{n-1}x_\mu}.\end{aligned}$$

Therefore, $-\widetilde{\sharp}_1(\mathrm{d}\mathcal{H}) + \mathbb{1}_n$ takes the expression

$$-\frac{1}{n}\frac{\partial H}{\partial y^i}\mathrm{d}^n x \otimes \frac{\partial}{\partial p^\mu_i} \wedge \frac{\partial^{n-1}}{\partial^{n-1}x_\mu} + \frac{\partial H}{\partial p^\mu_i}\mathrm{d}^n x \otimes \frac{\partial}{\partial y^i} \wedge \frac{\partial^{n-1}}{\partial^{n-1}x_\mu} + \mathrm{d}^n x \otimes \frac{\partial^n}{\partial^n x},$$

which is clearly semi-basic and satisfies $\iota_{-\widetilde{\sharp}_1(\mathrm{d}\mathcal{H})+\mathbb{1}_n}\mathrm{d}^n x = \mathrm{d}^n x$, proving that it is a Hamiltonian in the sense of Definition 4.1.1.

Now, since $-\widetilde{\sharp}_1(\mathrm{d}\mathcal{H})+\mathbb{1}_n$ is semi-basic over X, this allows us to define dynamics as follows.

Definition 4.1.2 (Hamilton-De Donder-Weyl equations for $\mathcal{H}$). We define the Hamilton–De Donder–Weyl equations for a section $\psi : X \to M$ as

$$\psi^*(\mathrm{d}\alpha) = \psi^*(\mathrm{d}\alpha + \{\alpha, \mathcal{H}\}) = \psi^*(\iota_{-\widetilde{\sharp}_1(\mathrm{d}\mathcal{H})+\mathbb{1}_n}\mathrm{d}\alpha)$$

for every $\alpha \in \Omega^{n-1}_H(M)$.

Remark 4.1.4. Note that, by definition of the Hamiltonian $\mathcal{H}$, we have that the form $\mathrm{d}\alpha + \{\alpha, \mathcal{H}\}$ is semi-basic over X. Hence, it may be interpreted as a map $M \to \bigwedge^n X$,

$$\begin{tikzcd} & \bigwedge^n M \arrow[d, phantom] \\ M \arrow[ur, "\mathrm{d}\alpha+\{\alpha,\mathcal{H}\}"] \arrow[r, dashed] & \bigwedge^n X \end{tikzcd},$$

so that we may write $\psi^*(\mathrm{d}\alpha + \{\alpha, \mathcal{H}\}) = (\mathrm{d}\alpha + \{\alpha, \mathcal{H}\}) \circ \psi$. In this way, the Hamilton–De Donder–Weyl equations read as

$$\psi^*(\mathrm{d}\alpha) = (\mathrm{d}\alpha + \{\alpha, \mathcal{H}\}) \circ \psi$$

for every $\alpha \in \Omega^{n-1}_H(M)$, which generalizes the equation $\frac{\mathrm{d}f}{\mathrm{d}t} = \frac{\partial f}{\partial t} + \{f, H\}$ from classical mechanics. From now on, we will write the equations in this way.

Remark 4.1.5. Note that the condition $\iota_{-\widetilde{\sharp}_1(\mathrm{d}\mathcal{H})+\mathbb{1}_n}\omega = \omega$ guarantees that for basic $(n-1)$-forms ω, we obtain $\mathrm{d}\omega + \{\omega, \mathcal{H}\} = \mathrm{d}\omega$, so that the evolution of basic forms is always determined, as it should be.

Example 4.1.2. Let us write the equations for the Hamiltonian of Example 4.1.1. Note that we have

$$\{\alpha, \mathcal{H}\} = (-1)^{\deg_H \mathcal{H}} \iota_{\widetilde{\sharp}_1(\mathrm{d}\mathcal{H})}\mathrm{d}\alpha = -\iota_{\widetilde{\sharp}_1(\mathrm{d}\mathcal{H})}\mathrm{d}\alpha.$$

Taking α to be $y^i \mathrm{d}^{n-1}x_\mu$ and $p_i^\mu \mathrm{d}^{n-1}x_\mu$ we obtain

$$\{y^i \mathrm{d}^{n-1}x_\mu, \mathcal{H}\} = \frac{\partial H}{\partial p_i^\mu}\mathrm{d}^n x - \mathrm{d}y^i \wedge \mathrm{d}^{n-1}x_\mu,$$

$$\{p_i^\mu \mathrm{d}^{n-1}x_\mu, \mathcal{H}\} = -\frac{\partial H}{\partial y^i}\mathrm{d}^n x - \mathrm{d}p_i^\mu \wedge \mathrm{d}^{n-1}x_\mu,$$

so that the equations determined by $\mathcal{H}$ for a section $\psi \colon X \to M$ given locally by $y^i(x)$ and $p_i^\mu(x)$ read

$$\frac{\partial y^i}{\partial x^\mu}\mathrm{d}^n x = \psi^*(\mathrm{d}y^i \wedge \mathrm{d}^{n-1}x_\mu) = (\mathrm{d}y^i \wedge \mathrm{d}^{n-1}x_\mu + \{y^i \mathrm{d}^{n-1}x_\mu, H\}) \circ \psi = \frac{\partial H}{\partial p_i^\mu}\mathrm{d}^n x$$

$$\frac{\partial p_i^\mu}{\partial x^\mu}\mathrm{d}^n x = \psi^*(\mathrm{d}p_i^\mu \wedge \mathrm{d}^{n-1}x_\mu) = (\mathrm{d}p_i^\mu \wedge \mathrm{d}^{n-1}x_\mu + \{p_i^\mu \mathrm{d}^{n-1}x_\mu, \mathcal{H}\}) \circ \psi = -\frac{\partial H}{\partial y^i}\mathrm{d}^n x,$$

recovering the well-known Hamilton–De Donder–Weyl equations of classical field theories.

In many cases, it will be interesting to have a linearized version of a solution. This corresponds to the following definition.

Definition 4.1.3 (Linear solution). A linear solution is a connection (given by a horizontal lift) on $\tau\colon M \to X$, h, such that

$$h^*(\mathrm{d}\alpha) = \mathrm{d}\alpha + \{\alpha, \mathcal{H}\}_{\widetilde{\sharp}_1}$$

for every $\alpha \in \Omega_H^{n-1}(M)$.

Regarding the extensions studied in Sec. 3.3, we have the following proposition.

Proposition 4.1.1. *Let $\mathcal{H}$ be a Hamiltonian on a regular graded Dirac manifold of order n, $(M, S^a, \sharp_a)$. Then, given an extension*

$$\widetilde{\sharp}_n : S^{n+1}[n] \to T^*M \otimes \mathrm{T}M/K_1$$

of $\sharp_n$ in such a way that $\widetilde{\sharp}_n(\mathrm{d}\mathcal{H})$ takes vertical values, we have that $-\widetilde{\sharp}_n(\mathrm{d}\mathcal{H}) + \mathbb{1}_1$ is semi-basic and that $\iota_{-\widetilde{\sharp}_n(\mathrm{d}\mathcal{H})+\mathbb{1}_1}\omega = \omega$ for every semi-basic 1-form ω.

Proof. Let $\alpha \in S^n$ and $u \in \bigvee_{n-1} M$. Then, since $-\widetilde{\sharp}_1(\mathrm{d}\mathcal{H}) + \mathbb{1}_n$ is semi-basic, $-\iota_{\widetilde{\sharp}_n(\mathrm{d}\mathcal{H})}\alpha + \alpha$ is semi-basic (since we have $\iota_{\widetilde{\sharp}_1(\mathrm{d}\mathcal{H})}\alpha = \iota_{\widetilde{\sharp}_n(\mathrm{d}\mathcal{H})}\alpha$, by the compatibility condition). Contracting by u preserves the semi-basic character, so that

$$\begin{aligned}\iota_u(-\iota_{\widetilde{\sharp}_n(\mathrm{d}\mathcal{H})}\alpha + \alpha) &= -\iota_u\iota_{\widetilde{\sharp}_n(\mathrm{d}\mathcal{H})}\alpha + \iota_u\alpha \\ &= -\iota_{\widetilde{\sharp}_n(\mathrm{d}\mathcal{H})}\iota_u\alpha + \iota_u\alpha + \iota_{\iota_{\widetilde{\sharp}_n(\mathrm{d}\mathcal{H})}u}\alpha\end{aligned}$$

is semi-basic. The proof will be finished once we show that the last term $\iota_{\iota_{\widetilde{\sharp}_1(\mathrm{d}\mathcal{H})}u}\alpha$ also is. Indeed, since $\sharp_1(\mathrm{d}\mathcal{H})$ takes vertical values, $\iota_{\widetilde{\sharp}_1(\mathrm{d}\mathcal{H})}u$ is at least 1-vertical and, together with the fact that α is at most $(n-1)$-horizontal, implies that $\iota_{\iota_{\widetilde{\sharp}_1(\mathrm{d}\mathcal{H})}u}\alpha$ is semi-basic, which finishes the proof. The fact that $\iota_{-\widetilde{\sharp}_n(\mathrm{d}\mathcal{H})+\mathbb{1}}\omega = \omega$ for every semi-basic 1-form ω follows from $\widetilde{\sharp}_n$ taking vertical values. □

We now deal with the problem of interpreting the possible extensions of the brackets in terms of the dynamics determined by a Hamiltonian.

Fix a Hamiltonian $\mathcal{H}$ and an extension $\widetilde{\sharp}_n$ that satisfies the hypotheses of Proposition 4.1.1. Then, we get a map

$$S^1 \to \tau^*(\mathrm{T}^* X), \quad \theta \mapsto \theta - \iota_{\widetilde{\sharp}_n(\mathrm{d}\mathcal{H})}\theta.$$

that is the identity on semi-basic forms. Taking the adjoint, we obtain a vector bundle map

$$\gamma_{\mathcal{H}} : \tau^*(\mathrm{T}X) \to \mathrm{T}M/K_1,$$

such that $\tau_* \circ \gamma_{\mathcal{H}} = \mathrm{id}_{\mathrm{T}X}$ (recall that the projection is well-defined modulo K_p). This map can be thought of as the horizontal lift of a connection "up to K_1". This connection will solve the Hamilton–De Donder–Weyl equations. We have the following result formalizing the previous idea.

Proposition 4.1.2. *Let h be the horizontal lift of an Ehresmann connection on $\tau\colon M \to X$. If its horizontal lift modulo K_1 coincides with $\gamma_{\mathcal{H}}$, we have that h solves the Hamilton–de Donder–Weyl equations determined by $\mathcal{H}$.*

Proof. Since the horizontal lift of h coincides with $\gamma_{\mathcal{H}}$ modulo K_1, we have $h^*\theta = \theta - \iota_{\widetilde{\sharp}_n(\mathrm{d}\mathcal{H})}\theta$ for every $\theta \in S^1$. Therefore, we can write

$$\begin{aligned} h^*(\theta_1 \wedge \cdots \wedge \theta_a) &= h^*\theta_1 \wedge \cdots \wedge h^*\theta_a \\ &= (\theta_1 - \iota_{\widetilde{\sharp}_n(\mathrm{d}\mathcal{H})}\theta_1) \wedge \cdots \wedge (\theta_a - \iota_{\widetilde{\sharp}_n(\mathrm{d}\mathcal{H})}\theta_a) \end{aligned}$$

for $\theta_1, \ldots, \theta_a \in S^1$. Hence, since $\widetilde{\sharp}_n$ takes vertical values, if every form $\theta_1, \ldots, \theta_a$ but possibly one is semi-basic, we may write

$$h^*(\theta_1 \wedge \cdots \wedge \theta_a) = \theta_1 \wedge \cdots \wedge \theta_a - \iota_{\widetilde{\sharp}_n(\mathrm{d}\mathcal{H})}(\theta_1 \wedge \cdots \wedge \theta_a).$$

In particular, letting $a = n$ and $\alpha \in S^n$ we have (employing the fact that $\widetilde{\sharp}_n$ is an extension) $h^*\alpha = \alpha - \iota_{\widetilde{\sharp}_n(\mathrm{d}\mathcal{H})}\alpha = \alpha - \iota_{\widetilde{\sharp}_1(\mathrm{d}\mathcal{H})}\alpha$, which clearly implies $h^*(\mathrm{d}\alpha) = \mathrm{d}\alpha + \{\alpha, \mathcal{H}\}$ for every $\alpha \in \Omega^{n-1}_H(M)$, finishing the proof. □

Notice that the previous computation works for any form $\alpha \in S^a$, and for arbitrary a, so that we have that suitable extensions of the brackets give a possible evolution of arbitrary forms, that is, if we denote by $\{\cdot,\cdot\}_{\widetilde{\sharp}_n}$ the induced bracket by the extension $\widetilde{\sharp}_n$ (see Eq. (10)), we have the following theorem.

Theorem 4.1.1. *Let $\widetilde{\sharp}_n$ be an extension that takes vertical values, $\mathcal{H} \in \Omega^n(M)[n]$ be a Hamiltonian and h be a horizontal lift of an Eheresmann connection whose vertical lift is $\gamma_{\mathcal{H}}$ modulo K_1. Then, h solves the Hamilton–De Donder–Weyl equations determined by $\mathcal{H}$ and we have*

$$h^*_{\mathcal{H}}(\mathrm{d}\alpha) = \mathrm{d}\alpha + \{\alpha, \mathcal{H}\}_{\widetilde{\sharp}_n}$$

for any $\alpha \in \Omega^{a-1}_H(M)$, with $1 \le a \le n$.

Proof. From the previous computation, for $\alpha \in \Omega^{a-1}_H(M)$ we have

$$h^*(\mathrm{d}\alpha) = \mathrm{d}\alpha - \iota_{\widetilde{\sharp}_n(\mathrm{d}\mathcal{H})}\mathrm{d}\alpha = \mathrm{d}\alpha + \{\alpha, \mathcal{H}\}_{\widetilde{\sharp}_n}.$$ □

We would like to end this discussion with the following result, which gives the relation between the possible extensions and the possible choices of solutions. Before stating and proving it first note that the space of Hamiltonians is affine, and so is the space of Ehresmann connections. Furthermore, if we are exclusively interested in studying the equations defined by Hamiltonians, we can restrict to those forms whose exterior derivative takes values in the following subbundle:

$$\Gamma := \{\alpha \in S^{n+1}[n] : -\widetilde{\sharp}_1(\alpha) + \mathbb{1}_n = \text{semi-basic}\}$$

and study extensions of the $\sharp$ maps defined only on Γ.

Theorem 4.1.2. *Suppose that, locally, $S^n = \langle \mathrm{d}\alpha_1, \dots, \mathrm{d}\alpha_k \rangle$, for certain Hamiltonian forms $\alpha_1, \dots, \alpha_k$. Then, there is a bijective correspondence between vertical valued extensions $\widetilde{\sharp}_n$ defined on Γ and the family of affine maps*

$$\gamma : \{\textit{Hamiltonians}\} \to \{\textit{Horizontal lifts modulo } K_1\},$$

satisfying the following two properties:

(i) *If $\mathrm{d}\mathcal{H}_1|_x = \mathrm{d}\mathcal{H}_2|_x$, then $\gamma_{\mathcal{H}_1}|_x = \gamma_{\mathcal{H}_2}|_x$.*
(ii) *If h is a connection such that its horizontal lift coincides with $\gamma(\mathcal{H})$ modulo K_1, then h solves the Hamilton–De Donder–Weyl equations.*

Proof. First, given the vertical extension of $\widetilde{\sharp}_n$, we obtain the corresponding map γ as before, that is, noticing that the map

$$\mathrm{d}f \mapsto \mathrm{d}f + \{f, \mathcal{H}\} = \mathrm{d}f - \iota_{\widetilde{\sharp}_n(\mathrm{d}\mathcal{H})}\mathrm{d}f$$

defines a map $S^1 \to S^1$ that takes values in semi-basic forms. By taking the adjoint we obtain the corresponding horizontal lift. Conversely, let γ be such a map and, for every Hamiltonian $\mathcal{H} \in \Omega^n_H(M)[n]$ define $\widetilde{\sharp}_n(\mathrm{d}\mathcal{H})$ to be the $(1,1)$ tensor field defined by

$$\iota_{\widetilde{\sharp}_n(\mathrm{d}\mathcal{H})}\mathrm{d}f = \mathrm{d}f - h^*\mathrm{d}f,$$

where h is the horizontal lift associated to the connection $\gamma(\mathcal{H})$. The first required property on γ implies that it defines an extension

$$\widetilde{\sharp}_n : \Gamma \to \mathrm{T}^* M \otimes \mathrm{T}M/K_1$$

and the fact that $\gamma(\mathrm{d}\mathcal{H})$ solves the Hamilton–De Donder–Weyl equations, together with the assumption that S^n is generated by the exterior differential of Hamiltonian forms, implies that

$$\widetilde{\sharp}_1 = \widetilde{\sharp}_n \wedge \mathbb{1}_{n-1},$$

so that it defines an extension of the brackets. □

4.2. *Differential forms with defined evolution*

We have shown in Sec. 4.1 that extensions of brackets define linear solutions for every possible choice of Hamiltonian in a way such that the expression $\mathrm{d}\alpha + \{\alpha, \mathcal{H}\}_{\widetilde{\sharp}_n}$ gives a possible evolution of the form α. Thus, if we identify the subfamily of Hamiltonian forms for which $\mathrm{d}\alpha + \{\alpha, \mathcal{H}\}_{\widetilde{\sharp}_n}$ is independent of the particular extension $\widetilde{\sharp}_n$, we are identifying the family of forms that have defined evolution, for every solution of Hamilton–De Donder–Weyl equations.

Theorem 4.2.1. *Let $(\tau : M \to X, S^a, \sharp_a)$ be a fibered graded Dirac manifold. Let $\mathcal{H} \in \Omega^n_H(M)[n]$ be a Hamiltonian and let $\gamma \in \Omega^{a-1}(M)$ be a form such that there* *exists a semi-basic form* $\beta \in \Omega^a(M)$ *with*

$$h^*(\mathrm{d}\gamma) = \beta$$

for every linear solution of Hamilton-De Donder-Weyl equations h*. Then,* $\gamma \in \Omega_H^{a-1}(M)$ *and it satisfies*:

$$\gamma \wedge \varepsilon \in \Omega_H^{n-1}(M)$$

for every basic closed form $\varepsilon \in \Omega^{n-a}(M)$*.*

Proof. Let us begin with the case $a = n$.

Lemma 4.2.1. *Let* $\gamma \in \Omega^{n-1}(M)$ *be an* $(n-1)$*-form and* $\beta \in \Omega^n(M)$ *be a semi-basic* n*-form such that* $h^*(\mathrm{d}\gamma) = \beta$ *for every solution of the Hamilton–de Donder–Weyl equations determined by a Hamiltonian* $\mathcal{H} \in \Omega_H^n(M)[n]$*. Then* $\gamma \in \Omega_H^{n-1}(M)$*.* □

Proof. The space of linear solutions is defined as those connections (given by a horizontal lift) h such that $h^*(\mathrm{d}\alpha) = \mathrm{d}\alpha + \{\alpha, \mathcal{H}\}$ for every $\alpha \in \Omega_H^{n-1}(M)$. Let us define, for each $\alpha \in S^n$,

$$\Phi_\alpha : \{\text{horizontal lifts}\} \to \bigwedge^n \mathrm{T}^* X, \quad h \mapsto h^*(\alpha) - (\alpha - \iota_{\widetilde{\sharp}_n(\mathrm{d}\mathcal{H})}\alpha).$$

Hence, if h is a horizontal lift such that $\Phi_\alpha(h) = 0$ for every $\alpha \in S^n$, we have that h solves the equations defined by $\mathcal{H}$. Also note that each Φ_α is an affine map. Now let $\gamma \in \Omega^{n-1}(M)$ and $\beta \in \Omega^n(M)$ be such that $h^*(\mathrm{d}\gamma) = \beta$ for every solution of the Hamilton–De Donder–Weyl equations. Define

$$\varphi : \{\text{horizontal lifts}\} \to \bigwedge^n \mathrm{T}^* X, \quad h \mapsto h^*(\mathrm{d}\alpha) - \beta.$$

Then φ is identically zero on $\bigcap_{\alpha \in S^a} \Phi_\alpha^{-1}(0)$. Since φ is affine and $\bigcap_{\alpha \in S^a} \Phi_\alpha^{-1}(0) \neq \varnothing$, we have that $\varphi \in \langle \Phi_\alpha : \alpha \in S^n \rangle$, which means that there is an $\alpha \in S^n$ such that $\varphi = \Phi_\alpha$. This immediately implies $\mathrm{d}\gamma = \alpha \in S^n$, proving that γ is Hamiltonian. □

Now, for arbitrary a, let $\gamma \in \Omega^{a-1}(M)$ and $\beta \in \Omega^a(M)$ semi-basic such that $h^*(\mathrm{d}\gamma) = \beta$ for every solution of the Hamilton–De Donder–Weyl equations. Let $\varepsilon \in \Omega^{n-a}(M)$ be closed and basic. Then,

$$h^*(\mathrm{d}(\gamma \wedge \varepsilon)) = \beta \wedge \varepsilon,$$

so that, by Lemma 4.2.1, $\gamma \wedge \varepsilon \in \Omega_H^{n-1}(M)$ for every closed and basic $(n-a)$-form ε. Hence, $\mathrm{d}\gamma \wedge \varepsilon \in S^n$ for every basic (not necessarily closed) $(n-a)$-form ε. The proof will be finished once we show that $\mathrm{d}\alpha \in S^a$, which will be a consequence of the following lemma.

Lemma 4.2.2. *Let* V *be a finite dimensional vector space, together with a surjective linear mapping* $\tau : V \to X$*, with* $\dim X = n$*. Let* $S^n \subseteq \bigwedge^n V^*$ *be a subspace* *consisting of $(n-1)$-horizontal forms and such that $\tau^*(\bigwedge^n X^*) \subseteq S^n$. If $\gamma \in \bigwedge^a V^*$ is such that $\gamma \wedge \varepsilon \in S^n$ for every basic ε, then $\gamma \in \langle \iota_{\bigwedge^{n-a} V} S^n \rangle$.*

Proof. Let e_μ, e_i denote a basis of V adapted to τ, so that $\tau(e_\mu) = e_\mu$ and $\tau(e_i) = 0$. Suppose

$$S^n = \langle \tau^*\omega, \alpha_1, \dots, \alpha_k \rangle,$$

where ω is the volume form on X satisfying $\omega(e_1, \dots, e_n) = 1$, and $\alpha_j = A^\mu_{ji}\omega_\mu \wedge e^i$, where $\omega_\mu = \iota_{e_\mu}\omega$. Without loss of generality, we can assume that γ takes the expression

$$\gamma = \gamma^{\mu_1,\dots,\mu_{n+1-a}} \omega_{\mu_1,\dots,\mu_{n+1-a}} \wedge e^i,$$

where $\omega_{\mu_1,\dots,\mu_{n+1-a}} = \iota_{e_{\mu_1}\wedge\dots\wedge e_{\mu_{n+1-a}}}\omega$. The condition that $\varepsilon \wedge \gamma \in S^n$ for each basic $(n-a)$ ε reads using the basis as $\gamma^{\mu_1,\dots,\mu_{n+1-a}}\omega_{\mu_{n+1-a}} \wedge e^i \in S^n$ which is equivalent to the existence of certain constants $B^{j,\mu_1,\dots,\mu_{n-a}}$ such that

$$\gamma^{\mu_1,\dots,\mu_{n+1-a}}\omega_{\mu_{n+1-a}} \wedge e^i = B^{j,\mu_1,\dots,\mu_{n-a}}\alpha_j = B^{j,\mu_1,\dots,\mu_{n-a}} A^{\mu_{n+1-a}}_{ji} \omega_{\mu_{n+1-a}} \wedge e^i.$$

Let $u^j := B^{j,\mu_1,\dots,\mu_{n-a}} e_{\mu_1} \wedge \dots \wedge e_{\mu_{n-a}}$. Then, the previous conditions read as $\gamma = \iota_{u^j}\alpha_j$, which finishes the proof. $\square$

In fact:

Theorem 4.2.2. *If $\alpha \in \Omega^{a-1}_H(M)$ is a Hamiltonian form such that*

$$\alpha \wedge \varepsilon \in \Omega^{n-1}_H(M)$$

for every closed basic $\varepsilon \in \Omega^{n-a}(M)$, then $\{\alpha, \mathcal{H}\}_{\widetilde{\sharp}_n}$ does not depend on the choice of $\widetilde{\sharp}_n$, for any Hamiltonian $\mathcal{H}$, if $\widetilde{\sharp}_n$ takes vertical values.

Proof. Observe that

$$\{\alpha \wedge \varepsilon, \mathcal{H}\}_{\widetilde{\sharp}_1} = \{\alpha \wedge \varepsilon, \mathcal{H}\}_{\widetilde{\sharp}_n} = \{\alpha, \mathcal{H}\}_{\widetilde{\sharp}_n} \wedge \varepsilon,$$

for any closed and basic $\varepsilon \in \Omega^{n-1-a}(M)$. Since an $(a+1)$-horizontal form is determined by its wedge product with all possible $(n-1-a)$-basic forms, we conclude that

$$\{\alpha, \mathcal{H}\}_{\widetilde{\sharp}_n}$$

is independent of the choice of $\widetilde{\sharp}_n$, as it only depends on $\widetilde{\sharp}_1$. $\square$

The previous two results motivate the following definition.

Definition 4.2.1 (Special Hamiltonian form). A Hamiltonian form $\alpha \in \Omega^{a-1}_H(M)$ is called special Hamiltonian if $\alpha \wedge \varepsilon \in \Omega^{n-1}_H(M)$ for every basic closed $(n-a)$-form ε. We denote by $\widetilde{\Omega}^{a-1}_H(M)$ the space of special Hamiltonian forms.

Remark 4.2.1. This gives a method of finding arbitrary $(a-1)$-horizontal a-forms that have a defined evolution. In particular, it allows us to look for forms that are closed on arbitrary solutions of the classical field equations, since we have clear candidates. We will apply this method in the following section.

To end this section, we prove that the space of special Hamiltonian forms defines a subalgebra.

Theorem 4.2.3. *Let $(\tau : M \to X, S^a, \sharp_a)$ be a fibered graded Dirac manifold. Then, special Hamiltonian forms define a graded subalgebra.*

Proof. The proof will follow from the following observation.

Lemma 4.2.3. *For each special Hamiltonian form $\alpha \in \widetilde{\Omega}^{a-1}_H(M)$ there is a multivector field $U_\alpha \in \mathfrak{X}^{n+1-a}(M)$ such that $\sharp_{a+b}(\mathrm{d}\alpha \wedge \varepsilon) = C(\alpha, \varepsilon)\iota_\varepsilon U_\alpha + K_{n+1-(a+b)}$, for each closed and basic b-form ε and for certain nonzero constant $C(\alpha, \varepsilon)$. The constant is chosen so that, for $b = 0$ (hence a function), we have that $\sharp_a(\mathrm{d}\alpha) = U_\alpha + K_{n+1-a}$.*

Proof. From the computations in Lemma 4.2.1, it is an straightforward computation to show the following. Let $\alpha \in \widetilde{\Omega}^{a-1}_H(M)$ be a special Hamiltonian form (hence Hamiltonian), and let $\alpha_1, \dots, \alpha_k$ denote local generators of S^n. Let B^j, $j = 1, \dots, k$ be the multivector fields built in said lemma, such that

$$\mathrm{d}\alpha = \iota_{B_j}\alpha_j.$$

Then, for every semi-basic b-form ε we have that there exists a nonzero number $C(a, b)$ such that $\mathrm{d}\alpha \wedge \varepsilon = C(a, b)\iota_{\iota_\varepsilon B^j}\alpha_j$.

Now, define, for this same α, the following map:

$$\bigwedge^{n-a} T^*X \to S^n \to TM, \qquad \varepsilon \mapsto \sharp_n(\mathrm{d}\alpha \wedge \varepsilon).$$

Taking the adjoint (see Remark 3.1.1), we obtain an element $\widehat{U}_\alpha \in \bigvee_{n-a} X \otimes TM$ that, by definition satisfies

$$\iota_\varepsilon \widehat{U}_\alpha = \sharp_n(\mathrm{d}\alpha \wedge \varepsilon),$$

where the contraction is understood in the first component. Since $\sharp$ takes vertical values in TM, if we define U_α to be the element obtained by exterior multiplication

$$\bigvee_{n-a} X \otimes TM \to \bigvee_{n+1-a} M/\{\text{2-vertical multivectors}\},$$

we still have $\iota_\varepsilon U_\alpha = \sharp_n(\mathrm{d}\alpha \wedge \varepsilon)$ for every basic $(n-a)$-form ε. Note here that contraction makes sense, as it is defined as well modulo 2-vertical multivector fields.

From the computation above we get

$$\sharp_n(\mathrm{d}\alpha \wedge \varepsilon) = C(a, n-a)\sharp_n(\alpha_j) \wedge (\iota_\varepsilon B^j),$$

so that

$$U_\alpha = (-1)^{n-a} C(a, n-a)\sharp_n(\alpha_j) \wedge B^j.$$

It is clear, using the relation above and Remark 2.1.7 (also that $\sharp_n(\alpha_j)$ is vertical), that this multivector satisfies the desired equality. In order to obtain the correct constant, we multiply by $C(a, n-a)^{-1}$, finishing the proof. $\square$

Now, let $\alpha \in \widetilde{\Omega}^a_H(M)$ and $\beta \in \widetilde{\Omega}^b_H(M)$ be special Hamiltonian forms. Let us show that $\{\alpha, \beta\} \wedge \varepsilon$ takes values in $\Omega^{n-1}_H(M)$ for every closed and basic ε of suitable degree. Using the multivector field from Lemma 4.2.3, notice that for a closed and basic 1-form ε we have the following:

$$\begin{aligned}
\{\alpha \wedge \varepsilon, \beta\} &= (-1)^{\deg_H \beta} \iota_{\sharp_{b+1}(\mathrm{d}\beta)}(\mathrm{d}\alpha \wedge \varepsilon) \\
&= (-1)^{\deg_H \beta}(\iota_{\sharp_{b+1}(\mathrm{d}\beta)}\mathrm{d}\alpha) \wedge \varepsilon + (-1)^{\deg_H \beta + \deg \alpha} \iota_{\iota_\varepsilon \sharp_{b+1}(\mathrm{d}\beta)}\mathrm{d}\alpha \\
&= \{\alpha, \beta\} \wedge \varepsilon + (-1)^{\deg \alpha} C(\beta, \varepsilon) \iota_{\sharp_{b+2}(\mathrm{d}\beta \wedge \varepsilon)}\mathrm{d}\alpha \\
&= \{\alpha, \beta\} \wedge \varepsilon - (-1)^{\deg_H \beta + \deg \alpha} C(\beta, \varepsilon)\{\alpha, \beta \wedge \varepsilon\}.
\end{aligned}$$

Note that, since both $\alpha \wedge \varepsilon$ and $\beta \wedge \varepsilon$ are Hamiltonian, and $C(\beta, \varepsilon)$ is nonzero, so is $\{\alpha, \beta\} \wedge \varepsilon$. Iterating this argument and using that both $\alpha \wedge \varepsilon_1 \wedge \cdots \wedge \varepsilon_j$ and $\alpha \wedge \varepsilon_1 \wedge \cdots \wedge \varepsilon_j$ are Hamiltonian, we conclude that $\{\alpha, \beta\} \wedge \varepsilon$ is Hamiltonian, for arbitrary closed and basic ε, which shows that special Hamiltonian forms define a subalgebra.

Remark 4.2.2. To give a complete picture, in the examples we will provide an explicit computation of the associated multivector field.

5. Applications to Classical Field Theories

Here we discuss several applications of the theory developed in the previous sections: the study of conserved quantities for regular and almost regular Lagrangians.

5.1. *Hamiltonian formalism for regular field theories*

Let us go through the construction of bracket extensions in the case of the reduced covariant formalism of classical field theories explained in Example 2.3.1. Recall that the graded Dirac structure is given by the map

$$\sharp_n(\mathrm{d}^n x) = 0, \quad \sharp_n(\mathrm{d}y^i \wedge \mathrm{d}^{n-1}x_\mu) = -\frac{\partial}{\partial p_i^\mu}, \quad \sharp_n(\mathrm{d}p_i^\mu \wedge \mathrm{d}^{n-1}x_\mu) = \frac{\partial}{\partial y^i}.$$

The $\sharp_1$ (modulo K_n) map is given by

$$\sharp_1(\mathrm{d}x^\mu) = K_n, \quad \sharp_1(\mathrm{d}y^i) = \frac{(-1)^n}{n}\frac{\partial}{\partial p_i^\mu} \wedge \frac{\partial^{n-1}}{\partial^{n-1}x_\mu} + K_n,$$

$$\sharp_1(\mathrm{d}p_i^\mu) = (-1)^{n-1}\frac{\partial}{\partial y^i} \wedge \frac{\partial^{n-1}}{\partial^{n-1}x_\mu} + K_n$$

and $\widetilde{\sharp}_1$ is obtained applying the anti-derivation rule, as mentioned in Remark 3.2.2. Let us determine the space $S^{n+1}[n]$, which will be the space on which the exterior derivative of Hamiltonians may take values.

Theorem 5.1.1. *$S^{n+1}[n]$ is generated by the following set of forms*:

$$\mathrm{d}y^i \wedge \mathrm{d}^n x, \quad \mathrm{d}p_i^\mu \wedge \mathrm{d}^n x, \quad \mathrm{d}p_i^\mu \wedge \mathrm{d}y^j \wedge \mathrm{d}^{n-1}x_\mu.$$

Proof. Recall that $S^{n+1}[n]$ is defined as the space of forms $\Theta \in \bigwedge^{n+1} M$ for which there exists $\widetilde{\sharp}_n(\Theta) \in \mathrm{T}^* M \otimes \mathrm{T}M$ with $\iota_{\widetilde{\sharp}_n(\Theta)}\alpha = \iota_{\widetilde{\sharp}_1(\Theta)}\alpha = \iota_{\widetilde{\sharp}_n(\alpha)}\Theta$ for every $\alpha \in S^n$. Note that α is $(n-1)$-horizontal, so that $\iota_{\widetilde{\sharp}_1(\Theta)}\alpha$ is $(n-2)$-horizontal. Therefore, $\iota_{\widetilde{\sharp}_n(\alpha)}\Theta$ is $(n-2)$-horizontal and, since $\sharp_n(\alpha)$ is vertical, Θ must be $(n-2)$-horizontal. In Table 1, the reader may find represented all possible values of $\iota_{\sharp_n(\alpha)}\Theta$ for all possible candidates $\Theta \in \bigwedge^{n+1} M$.

An elementary computation shows that any linear combination of the forms on the left, Θ, for which there is a $(1,1)$-tensor, $\widetilde{\sharp}_n(\Theta)$ satisfying $\iota_{\widetilde{\sharp}_n(\Theta)}\alpha = \iota_{\sharp_n(\alpha)}\Theta$, must have null coefficients on the last two elements ($\mathrm{d}y^j \wedge \mathrm{d}y^k \wedge \mathrm{d}^{n-1}x_\alpha$ and $\mathrm{d}p_j^\alpha \wedge \mathrm{d}p_k^\beta \wedge \mathrm{d}^{n-1}x_\gamma$). Furthermore, the first two elements are clearly in $S^{n+1}[n]$, as we may choose

$$\widetilde{\sharp}_n(\mathrm{d}y^j \wedge \mathrm{d}^n x) := \frac{1}{n}\mathrm{d}x^\mu \otimes \frac{\partial}{\partial p_i^\mu}, \quad \widetilde{\sharp}_n(\mathrm{d}p_j^\alpha \wedge \mathrm{d}^n x) := \mathrm{d}x^\alpha \otimes \frac{\partial}{\partial y^j}.$$

Table 1. Values of $\iota_{\sharp_n(\alpha)}\Theta$, for $(n+1)$-forms Θ (on the left) and forms $\alpha \in S^n$ (on top).

	$\mathrm{d}y^i \wedge \mathrm{d}^{n-1}x_\mu$	$\mathrm{d}p_i^\mu \wedge \mathrm{d}^{n-1}x_\mu$	$\mathrm{d}^n x$
$\mathrm{d}y^j \wedge \mathrm{d}^n x$	0	$\delta_i^j \mathrm{d}^n x$	0
$\mathrm{d}p_j^\alpha \wedge \mathrm{d}^n x$	$\delta_\mu^\alpha \delta_j^i \mathrm{d}^n x$	0	0
$\mathrm{d}y^j \wedge \mathrm{d}p_k^\alpha \wedge \mathrm{d}^{n-1}x_\beta$	$\delta_\mu^\alpha \delta_k^i \mathrm{d}y^j \wedge \mathrm{d}^{n-1}x_\beta$	$\delta_i^j \mathrm{d}p_k^\alpha \wedge \mathrm{d}^{n-1}x_\beta$	0
$\mathrm{d}y^j \wedge \mathrm{d}y^k \wedge \mathrm{d}^{n-1}x_\alpha$	0	$\delta_i^j \mathrm{d}y^k \wedge \mathrm{d}^{n-1}x_\alpha$ $-\delta_i^k \mathrm{d}y^j \wedge \mathrm{d}^{n-1}x_\alpha$	0
$\mathrm{d}p_j^\alpha \wedge \mathrm{d}p_k^\beta \wedge \mathrm{d}^{n-1}x_\gamma$	$\delta_\mu^\alpha \delta_j^i \mathrm{d}p_k^\beta \wedge \mathrm{d}^{n-1}x_\gamma$ $-\delta_\mu^\beta \delta_k^i \mathrm{d}p_j^k \wedge \mathrm{d}^{n-1}x_\gamma$	0	0

Hence, it only remains to study which possible linear combinations $\Theta = A^{k\beta}_{j\alpha}\mathrm{d}y^j \wedge \mathrm{d}p^\alpha_k \wedge \mathrm{d}^{n-1}x_\beta$ belong to $S^{n+1}[n]$. We look for a $(1,1)$-tensor field

$$\begin{aligned}\widetilde{\sharp}_n(\Theta) = \mathrm{d}x^\mu \otimes \left(C^\nu_\mu \frac{\partial}{\partial x^\nu} + D^j_\mu \frac{\partial}{\partial y^j} + E^\nu_{\mu j} \frac{\partial}{\partial p^\nu_j} \right) \\ + \mathrm{d}y^i \otimes \left(C^\nu_i \frac{\partial}{\partial x^\nu} + D^j_i \frac{\partial}{\partial y^j} + E^\nu_{ij} \frac{\partial}{\partial p^\nu_j} \right) \\ + \mathrm{d}p^\mu_i \otimes \left(C^{i\nu}_\mu \frac{\partial}{\partial x^\nu} + D^{ij}_\mu \frac{\partial}{\partial y^j} + E^{i\nu}_{\mu j} \frac{\partial}{\partial p^\nu_j} \right)\end{aligned}$$

satisfying $\iota_{\widetilde{\sharp}_1(\Theta)}\alpha = \iota_{\sharp_n(\alpha)}\Theta$ for every $\alpha \in S^n$. We have:

(i) For $\alpha = \mathrm{d}^n x$, we have $\iota_{\sharp_n(\alpha)}\Theta = 0$, which implies $C^\mu_\mu = C^\nu_i = C^{i\nu}_\mu = 0$.

(ii) For $\alpha = \mathrm{d}y^i \wedge \mathrm{d}^{n-1}x_\mu$, we have $\iota_{\sharp_n(\alpha)}\Theta = -A^{i\beta}_{j\mu}\mathrm{d}y^j \wedge \mathrm{d}^{n-1}x_\mu$, which implies that $C^\mu_\nu = 0$ and

$$-A^{i\beta}_{j\mu}\mathrm{d}y^j \wedge \mathrm{d}^{n-1}x_\beta = D^i_\mu \mathrm{d}^n x + D^i_j \mathrm{d}y^j \wedge \mathrm{d}^{n-1}x_\mu + D^{ji}_\nu \mathrm{d}p^\nu_j \wedge \mathrm{d}^{n-1}x_\mu,$$

from which we obtain $D^i_\mu = 0$, $D^{ji}_\nu = 0$ and $-A^{i\beta}_{j\mu} = \delta^\beta_\mu D^i_j$ and hence $\Theta = -D^k_j \mathrm{d}y^j \wedge \mathrm{d}p^\alpha_k \wedge \mathrm{d}^{n-1}x_\alpha$.

It only remains to show that we can define $\widetilde{\sharp}_n(\Theta)$ for $\Theta = \mathrm{d}p^\mu_i \wedge \mathrm{d}y^j \wedge \mathrm{d}^{n-1}x_\mu$. Indeed, notice that we may choose

$$\widetilde{\sharp}_n(\Theta) := \mathrm{d}y^j \otimes \frac{\partial}{\partial y^k} + \mathrm{d}p^\mu_k \otimes \frac{\partial}{\partial p^\mu_k},$$

finishing the proof. □

Now let us look at the possible expression of a Hamiltonian (Definition 4.1.1).

Theorem 5.1.2. *Let $\mathcal{H} \in \Omega^n_H(M)[n]$ be a Hamiltonian. Then, up to a closed term it has the following local expression:*

$$\mathcal{H} = H\mathrm{d}^n x - p^\mu_i \mathrm{d}y^i \wedge \mathrm{d}^{n-1}x_\mu$$

for certain function $H(x^\mu, y^i, p^\mu_i)$.

Proof. Recall that $\widetilde{\sharp}_1(\mathrm{d}\mathcal{H}) + \mathbb{1}_n$ has to be a semi-basic form. A quick computation shows that this implies

$$\mathrm{d}\mathcal{H} = A_i \mathrm{d}y^i \wedge \mathrm{d}^n x + B^i_\mu \mathrm{d}p^\mu_i \wedge \mathrm{d}^n x - \mathrm{d}p^\mu_i \wedge \mathrm{d}y^i \wedge \mathrm{d}^{n-1}x_\mu$$

for certain functions A_i and B^i_μ. Closedness of the previous form implies that there is a function H such that $A_i = \frac{\partial H}{\partial y^i}$ and $B^i_\mu = \frac{\partial H}{\partial p^\mu_i}$, which finishes the proof. □

Finally, let us study the algebra of special Hamiltonian forms.

Theorem 5.1.3. *With the canonical graded Poisson structure on* $M = \bigwedge_2^n Y / \bigwedge_1^n Y$ *special Hamiltonian forms close an subalgebra and:*

(i) *For* $a = 1, \ldots, n-1$, *if* $\alpha \in \widetilde{\Omega}_H^{a-1}(M)$, *then*

$$\mathrm{d}\alpha \in \langle \mathrm{d}^a x_{\widetilde{\mu}}, \mathrm{d}y^i \wedge \mathrm{d}^{a-1} x_{\widetilde{\nu}} \colon |\widetilde{\mu}| = n-a, |\widetilde{\nu}| = n-a-1 \rangle.$$

(ii) *For* $a = n$, *we have* $\widetilde{\Omega}_H^{n-1}(M) = \Omega_H^{n-1}(M)$.

Proof. The second claim is clear. It only remains to show that if $a = 1, \ldots, n-1$ and $\alpha \in \widetilde{\Omega}_H^{a-1}(M)$, then $\mathrm{d}\alpha$ is necessarily in $\langle \mathrm{d}^a x_{\widetilde{\mu}}, \mathrm{d}y^i \wedge \mathrm{d}^{a-1} x_{\widetilde{\nu}} : |\widetilde{\mu}| = n-a, |\widetilde{\nu}| = n-a-1 \rangle$. Suppose that

$$\mathrm{d}\alpha = A^{\widetilde{\mu}} \mathrm{d}^a x_{\widetilde{\mu}} + A_i^{\widetilde{\mu}} \mathrm{d}y^i \wedge \mathrm{d}^{a-1} x_{\widetilde{\mu}} + A^{i\widetilde{\mu}} \mathrm{d}p_i^\mu \wedge \mathrm{d}^{a-1} x_{\widetilde{\mu}\mu}.$$

Imposing the condition of being special Hamiltonian with $\varepsilon = \mathrm{d}^{n-a} x^{\widetilde{\nu}}$, we get that

$$\begin{aligned}\varepsilon \wedge \mathrm{d}\alpha = {} & A^{\widetilde{\nu}} \mathrm{d}^n x + (-1)^{n-a} A_i^{\widetilde{\mu}} \delta_{\widetilde{\mu}}^{\widetilde{\nu}\mu} \mathrm{d}y^i \wedge \mathrm{d}^{n-1} x_\mu \\ & + (-1)^{n-1} A^{i\widetilde{\mu}} \delta_{\widetilde{\mu}\mu}^{\widetilde{\nu}\nu} \mathrm{d}p_i^\mu \wedge \mathrm{d}^{n-1} x_\nu \in S^n.\end{aligned}$$

The first two terms clearly satisfy this condition. The last one can be expanded as

$$(-1)^{n-a} A^{i\widetilde{\mu}} \delta_{\widetilde{\mu}}^{\widetilde{\nu}} \mathrm{d}p_i^\mu \wedge \mathrm{d}^{n-1} x_\mu + (-1)^{n-a} \sum_{\mu \neq \nu} A^{i\widetilde{\mu}} \delta_{\widetilde{\mu}\mu}^{\widetilde{\nu}\nu} \mathrm{d}p_i^\mu \wedge \mathrm{d}^{n-1} x_\nu$$

and since $a < n$, we can choose $\widetilde{\nu}$ so that $\delta_{\widetilde{\mu}}^{\widetilde{\nu}} = 0$ so we obtain $A^{i\widetilde{\mu}} = 0$, proving the result. By Theorem 4.2.3, they determine a subalgebra. □

Remark 5.1.1. In order to give a more complete picture, we would like to explicitly find the associated multivectors from Lemma 4.2.3. To the form $\mathrm{d}y^i \wedge \mathrm{d}^{a-1} x_{\mu_1, \ldots, \mu_{n+1-a}}$, we assign the multivector

$$U = \frac{1}{n+1-a} \sum_{j=1}^{n+1-a} (-1)^{j-1} \frac{\partial}{\partial p_i^{\mu_j}} \wedge \frac{\partial^{n-a}}{\partial^{n-a} x^{\mu_1, \ldots, \widehat{\mu_j}, \ldots, \mu_{n+1-a}}},$$

which satisfies the required property. Indeed, contracting (without loss of generality) by $\varepsilon = \mathrm{d}x^{\mu_1}$ on U gives

$$\iota_\varepsilon U = \frac{1}{n+1-a} \sum_{j=2}^{n+1-a} (-1)^j \frac{\partial}{\partial p_i^{\mu_j}} \wedge \frac{\partial^{n-a}}{\partial^{n-a} x^{\mu_2, \ldots, \widehat{\mu_j}, \ldots, \mu_{n+1-a}}},$$

which is a Hamiltonian multivector field for $\varepsilon \wedge \mathrm{d}y^i \wedge \mathrm{d}^{a-1} x_{\mu_1, \ldots, \mu_{n+1-a}}$ up to a constant.

5.2. *Bracket formalism for almost regular Lagrangians: Yang–Mills theories*

Let $\pi\colon Y \to X$ denote a fibered manifold over an n-dimensional base with coordinates (x^μ, y^i), and let $\mathcal{L} : J^1Y \to \bigwedge^n X$ be a first-order Lagrangian density (for higher order theories we refer to [16]), locally given by $\mathcal{L} = L(x^\mu, y^i, z^i_\mu)\mathrm{d}^n x$. The multisymplectic formalism is obtained by pullbacking the natural multisymplectic form on $\bigwedge^n_2 Y$ by the Legendre transformation, $\mathrm{Leg}_{\mathcal{L}} : J^1\pi \to \bigwedge^n_2 Y$, which is locally given by

$$\mathrm{Leg}^*_{\mathcal{L}}\, p = L - \frac{\partial L}{\partial z^i_\mu} z^i_\mu, \quad \mathrm{Leg}^*_{\mathcal{L}}\, p^\mu_i = \frac{\partial L}{\partial z^\mu_i}.$$

The reduced Legendre transformation is obtained by composing with the projection $\bigwedge^n_2 Y \to \bigwedge^n_2 Y / \bigwedge^n_1 Y$,

$$\begin{tikzcd} J^1\pi \arrow[rr, "\mathrm{Leg}_{\mathcal{L}}"] \arrow[drr, bend right, "\mathrm{leg}_{\mathcal{L}}"'] & & \bigwedge^n_2 Y \arrow[d] \\ & & \bigwedge^n_2 Y / \bigwedge^n_1 Y \end{tikzcd}.$$

When the reduced Legendre transformation defines a local diffeomorphism, we say that $\mathcal{L}$ is a *regular* Lagrangian. In this case, locally, we are dealing with the case treated in the previous subsection. However, a great deal of field theories (defined by the corresponding Lagrangian density) are not regular. Nevertheless, when the image $\mathrm{leg}_{\mathcal{L}}\left(J^1\pi\right)$ is a submanifold for which $\mathrm{leg}_{\mathcal{L}}\colon J^1\pi \to \mathrm{Im}\,\mathrm{leg}_{\mathcal{L}}$ defines a submersion, then we may apply the constructions of this paper:

(i) Use the construction of Theorem 2.2.1 to induce a graded Dirac structure on $\mathrm{Im}\,\mathrm{leg}_{\mathcal{L}}$ from the graded Dirac structure on $\bigwedge^n_2 Y / \bigwedge^n_1 Y$. This first step defines a bracket theory on the constraint submanifold.

(ii) Let $h\colon \mathrm{Im}\,\mathrm{leg}_{\mathcal{L}} \to \mathrm{Im}\,\mathrm{Leg}_{\mathcal{L}}$ be the diffeomorphism given by the inverse of the natural projection $\mathrm{Im}\,\mathrm{Leg}_{\mathcal{L}} \to \mathrm{Im}\,\mathrm{leg}_{\mathcal{L}}$.

(iii) Set $\mathcal{H} := h^*\Theta$ as the Hamiltonian, and study the dynamics employing the techniques developed above: existence of of special Hamiltonian forms and conserved quantities of arbitrary degree.

In order to illustrate this, let us study the particular case of Yang–Mills theories, where the Lagrangian is almost regular.

Let X be an n-dimensional pseudo-Riemannian manifold, and let $P \to X$ be a principal bundle with structure group G acting freely and transitively on the fibers on the right. Suppose there is an Ad-equivariant non-degenerate bilinear form $\langle\cdot,\cdot\rangle$ on $\mathfrak{g}$. Recall that a principal connection on P is an Ehresmann connection that is G invariant, namely a smooth choice of horizontal subspace $p \mapsto H_p$ satisfying $(R_g)_* H_p = H_{p\cdot g}$. Such connections can be identified with connection 1-forms

$\theta \in \Omega^1(P, \mathfrak{g})$, namely connections satisfying

(i) For every $\xi \in \mathfrak{g}$, $\theta(\widetilde{\xi}) = \xi$, where $\widetilde{\xi}$ denotes the fundamental vector field associated to the right action of G on P.
(ii) For every $g \in G$, $(R_g)^*\theta = \mathrm{Ad}_{g^{-1}} \cdot\theta$.

The curvature of θ is given by the $\mathfrak{g}$-valued 2-form $\omega_\theta = D_\theta\, \theta$, where $(D_\theta\theta)(X, Y) = (\mathrm{d}\theta)(X^h, Y^h)$, X^h and Y^h denoting the horizontal components of X and Y, respectively, defined by the connection. Since ω_θ is a semi-basic 2-form on P, and it satisfies $(R_g)^*\omega_\theta = \mathrm{Ad}_{g^{-1}} \cdot\omega_\theta$, it induces a 2-form $\widetilde{\omega}_\theta \in \Omega^2(M, \mathrm{Ad}\, P)$, where $\mathrm{Ad}_P = P \times_G \mathfrak{g}$ is the associated bundle to the adjoint representation of G. Since the chosen metric on $\mathfrak{g}$ is Ad-equivariant, it induces a non-degenerate bilinear form on each fiber of Ad_P, which we denote by $\langle\cdot,\cdot\rangle$ as well. This metric, together with the pseudo-Riemannian structure on M, allows us to introduce a scalar quantity

$$-\frac{1}{4}\langle\omega_\theta, \omega_\theta\rangle.$$

Then one tries to solve the variational problem on the space of principal connection given by the following action:

$$-\frac{1}{4}\int_X \langle\omega_\theta, \omega_\theta\rangle\mu,$$

where μ denotes the canonical volume form on X induced by the pseudo-Riemannian metric.

In our setting, principal connections are given by the sections of the bundle of principal connections: $Y = J^1(P \to M)/G \to X$ and the Lagrangian is given by

$$\mathcal{L} : J^1Y \to \bigwedge^n X \qquad (j^1\theta) \mapsto -\frac{1}{4}\langle\omega_\theta, \omega_\theta\rangle\mu.$$

If we have a local trivialization over an open subset $U \subseteq X$ given by certain section $s : U \to P$, we have a natural identification

$$Y|_U = J^1(P \to X)/G|_U \cong \mathrm{T}^*U \otimes \mathfrak{g}$$

given by $\theta \mapsto s^*\theta$. Then, if e_i denotes a basis on $\mathfrak{g}$ and (x^μ) denotes coordinates on U, we have induced coordinates on $Y|_U$ given by (x^μ, A^i_μ) representing the 1-form

$$\theta = A^i_\mu \mathrm{d}x^\mu \otimes e_a.$$

With this identification, J^1Ycarries natural coordinates $(x^\mu, A^i_\mu, A^i_{\mu,\nu})$. Then, the curvature reads

$$\omega_\theta = F^i_{\mu\nu}\mathrm{d}x^\mu \wedge \mathrm{d}x^\nu \otimes e_i = \left(\frac{\partial A^i_\nu}{\partial x^\mu} - \frac{\partial A^i_\mu}{\partial x^\nu} + f^i_{jk}A^j_\mu A^k_\nu\right)\mathrm{d}x^\mu \wedge \mathrm{d}x^\nu \otimes e_i,$$

where f^i_{jk} are the structure constants of the Lie algebra $\mathfrak{g}$, namely $[e_j, e_k] = f^i_{jk} e_i$. Then, the Lagrangian is locally given by

$$\mathcal{L} = -\frac{1}{4} F^{\mu\nu}_i F^i_{\mu\nu} \sqrt{|g|} \mathrm{d}^n x.$$

An elementary computation shows that the Legendre transformation is given by

$$\mathrm{Leg}^*_{\mathcal{L}}\, p^{\mu\nu}_i = \frac{\partial L}{\partial A^i_{\mu,\nu}} = F^{\mu\nu}_i \sqrt{|g|},$$

$$\mathrm{Leg}^*_{\mathcal{L}}\, p = \frac{\partial L}{\partial A^i_{\mu,\nu}} A^i_{\mu,\nu} - L = \left(-\frac{1}{4} F^{\mu\nu}_i F^i_{\mu\nu} + \frac{1}{2} f^i_{jk} F^{\mu\nu}_i A^j_\nu A^k_\mu \right) \sqrt{|g|}$$

so that it defines the constraint submanifold $N = \{p^{\mu\nu}_i + p^{\nu\mu}_i = 0\}$ on $\bigwedge^n_2 Y$. We refer to [45, 46] for more information regarding the multisymplectic treatment of gauge theories.

Let us compute the induced graded Dirac structure on the image of the Legendre transformation, as well as the induced Hamiltonian and the algebra of special Hamiltonian forms.

5.2.1. *The induced graded Dirac structure on* $N = \mathrm{Im}\,\mathrm{Leg}_{\mathcal{L}}$

As we know from Example 2.3.1, the graded Dirac structure on $\bigwedge^n_2 Y$ is given by $S^n = \langle \mathrm{d}^n x, \mathrm{d}A^i_\mu \wedge \mathrm{d}^{n-1} x_\nu, \mathrm{d}p^{\mu\nu}_i \wedge \mathrm{d}^{n-1} x_\nu \rangle$ with

$$\sharp_n(\mathrm{d}^n x) = 0, \quad \sharp_n(\mathrm{d}A^i_\mu \wedge \mathrm{d}^{n-1} x_\nu) = -\frac{\partial}{\partial p^{\mu\nu}_a}, \quad \sharp_n(\mathrm{d}p^{\mu\nu}_i \wedge \mathrm{d}^{n-1} x_\nu) = \frac{\partial}{\partial A^i_\mu}.$$

Recall that the pullback of a graded Dirac structure is given by the following steps:

(i) Identify those forms $\alpha \in S^n$ with $\sharp_n(\alpha)$ tangent to N,
(ii) Define $\widetilde{S}^n$ to be generated by the pullback of said forms and let $\widetilde{\sharp}_n(\alpha) := \sharp_n(\alpha)$.

In our case the family of forms $\alpha \in S^n$ such that $\sharp_n(\alpha)$ is tangent to the submanifold defined by the equations $p^{\mu\nu}_i + p^{\nu\mu}_i = 0$ is generated by the following forms:

$$\mathrm{d}^n x, \quad \mathrm{d}A^i_\mu \wedge \mathrm{d}^{n-1} x_\nu - \mathrm{d}A^i_\nu \wedge \mathrm{d}^{n-1} x_\mu, \quad \mathrm{d}p^{\mu\nu}_i \wedge \mathrm{d}^{n-1} x_\nu.$$

Let us introduce coordinates $(x^\mu, A^i_\mu, \widetilde{p}^{\mu\nu}_i)$ on N, with $\widetilde{p}^{\mu\nu}_i = p^{\mu\nu}_i$, if $\mu < \nu$ and $\widetilde{p}^{\mu\nu}_i = -p^{\mu\nu}_i$, if $\mu > \nu$, so that

$$\frac{\partial}{\partial \widetilde{p}^{\mu\nu}_i} = \frac{\partial}{\partial p^{\mu\nu}_i} - \frac{\partial}{\partial p^{\nu\mu}_i},$$

for $\mu < \nu$. Then, the graded Dirac structure on N is locally given by $\widetilde{S}^n = \langle \mathrm{d}^n x, \mathrm{d}A^i_\mu \wedge \mathrm{d}^{n-1} x_\nu - \mathrm{d}A^i_\nu \wedge \mathrm{d}^{n-1} x_\mu, \mathrm{d}\widetilde{p}^{\mu\nu}_i \mathrm{d}^{n-1} x_\nu - \mathrm{d}\widetilde{p}^{\nu\mu}_i \wedge \mathrm{d}^{n-1} x_\mu \rangle$, where sum in the last term is assumed to happen for pairs of indices such that $\mu < \nu$ (and $\nu < \mu$). Let us make an abuse notation by writing

$$\mathrm{d}\widetilde{p}^{\mu\nu}_i \wedge \mathrm{d}^{n-1} x_\nu - \mathrm{d}\widetilde{p}^{\nu\mu}_a \wedge \mathrm{d}^{n-1} x_\mu = \mathrm{d}\widetilde{p}^{\mu\nu}_i \wedge \mathrm{d}^{n-1} x_\mu$$

and assuming that $\widetilde{p}_i^{\nu\mu} = -\widetilde{p}_i^{\mu\nu}$. Then, the $\widetilde{\sharp}_n$ maps are given by

$$\widetilde{\sharp}_n(\mathrm{d}^n x) = 0, \quad \widetilde{\sharp}_n(\mathrm{d}A^i_\mu \wedge \mathrm{d}^{n-1}x_\nu - \mathrm{d}A^i_\nu \wedge \mathrm{d}^{n-1}x_\mu) = -\frac{\partial}{\partial \widetilde{p}_i^{\mu\nu}},$$

$$\widetilde{\sharp}_n(\mathrm{d}\widetilde{p}_i^{\mu\nu} \wedge \mathrm{d}^{n-1}x_\nu) = \frac{\partial}{\partial A^i_\mu}.$$

5.2.2. *Special Hamiltonian forms and the Hamiltonian*

From the computations of the previous subsection, we conclude that the Hamiltonian is

$$\mathcal{H} = \left(-\frac{1}{4}\widetilde{p}_i^{\mu\nu}\widetilde{p}^i_{\mu\nu} + \frac{1}{2}f^i_{jk}\widetilde{p}_i^{\mu\nu}A^j_\nu A^k_\mu\right)\mathrm{d}^n x - \widetilde{p}_i^{\mu\nu}\mathrm{d}A^i_\mu \wedge \mathrm{d}^{n-1}x_\nu$$

and thus the equations of motion in coordinates read as

(i) Taking $\alpha = A^i_\mu \mathrm{d}^{n-1}x_\nu - A^i_\nu \mathrm{d}^{n-1}x_\mu$ we obtain

$$\left(\frac{\partial A^i_\mu}{\partial x^\nu} - \frac{\partial A^i_\nu}{\partial x^\mu}\right)\mathrm{d}^n x = \psi^*(\mathrm{d}\alpha) = \mathrm{d}\alpha + \{\alpha, \mathcal{H}\} = \left(-\widetilde{p}^i_{\mu\nu} + f^i_{jk}A^j_\mu A^k_\nu\right)\mathrm{d}^n x.$$

(ii) Taking $\alpha = \widetilde{p}_i^{\mu\nu}\mathrm{d}^{n-1}x_{nu}$ we obtain

$$\frac{\partial \widetilde{p}_i^{\mu\nu}}{\partial x^\mu}\mathrm{d}^n x = \mathrm{d}\alpha + \{\alpha, \mathcal{H}\} = -f^j_{ik}\widetilde{p}_j^{\mu\nu}A^j_k \mathrm{d}^n x,$$

which are the well-known Yang–Mills equations.

Let us now focus on obtaining (what we will show to be) the subalgebra of special Hamiltonian forms, by determining what values can their exterior derivatives take.

Theorem 5.2.1. *Special Hamiltonian forms close a subalgebra and their exterior differentials are generated by:*

(i) *For* $1 \le a \le n-2$ *:*

$$\sum_{j=1}^{a}(-1)^{j-1}\mathrm{d}A^i_{\nu_j} \wedge \mathrm{d}^{a-1}x_{\nu_1,\ldots,\widehat{\nu_j},\ldots,\nu_a}, \quad \mathrm{d}^{b+1}x_{\mu_1\cdots\mu_{n-b}}.$$

(ii) *For* $a = n-1$ *:*

$$\mathrm{d}A^i_\alpha \wedge \mathrm{d}^{n-1}x_{\beta\gamma} + \mathrm{d}A^i_\gamma \wedge \mathrm{d}^{n-1}x_{\alpha\beta} + \mathrm{d}A^i_\beta \wedge \mathrm{d}^{n-1}x_{\gamma\alpha},$$

$$\mathrm{d}^{n-1}x_\mu, \quad \mathrm{d}\widetilde{p}_i^{\mu\nu} \wedge \mathrm{d}^{n-1}x_{\mu\nu}.$$

(iii) *For* $a = n$ *: by the whole* $\widetilde{S}^n$.

Proof. The last item is clear. Suppose $\alpha \in \widetilde{\Omega}^{a-1}_H(M)$, for $1 \le a \le n-1$ and write

$$\mathrm{d}\alpha = C^{\mu\widetilde{\mu}}\mathrm{d}A^i_\mu \wedge \mathrm{d}^{a-1}x_{\widetilde{\mu}} + D^{i\widetilde{\mu}}_{\mu\nu}\mathrm{d}\widetilde{p}_i^{\mu\nu} \wedge \mathrm{d}^{a-1}x_{\widetilde{\mu}}.$$

Let $\varepsilon = \mathrm{d}^{n-a}x^{\widetilde{\nu}}$. Then, since α is special Hamiltonian we must have

$$\varepsilon \wedge \mathrm{d}\alpha =(-1)^{n-a}C^{\mu\widetilde{\mu}}\delta^{\widetilde{\nu}\alpha}_{\widetilde{\mu}}\mathrm{d}A^i_\mu \wedge \mathrm{d}^{n-1}x_\alpha + (-1)^{n-a}D^{i\widetilde{\mu}}_{\mu\nu}\delta^{\widetilde{\nu}\alpha}_{\widetilde{\mu}}\mathrm{d}\widetilde{p}^{\mu\nu}_i \wedge \mathrm{d}^{n-1}x_\alpha \in S^n.$$

If $\varepsilon \wedge \mathrm{d}\alpha \in S^n$, we necessarily have that $\iota_U(\varepsilon \wedge \mathrm{d}\alpha) = 0$ for every $U \in (S^n)^{\circ,n}$. Hence, taking

$$U = \frac{\partial}{\partial A^i_\alpha} \wedge \frac{\partial^{n-1}}{\partial^{n-1}x_\beta} + \frac{\partial}{\partial A^i_\beta} \wedge \frac{\partial^{n-1}}{\partial^{n-1}x_\alpha}$$

and imposing $\iota_U(\varepsilon \wedge \mathrm{d}\alpha) = 0$ we get that $C^{\alpha\widetilde{\mu}}_i\delta^{\widetilde{\nu}\beta}_{\widetilde{\mu}} + C^{\beta\widetilde{\mu}}_i\delta^{\widetilde{\nu}\alpha}_{\widetilde{\mu}} = 0$. Taking $\alpha = \mu_0$, $\widetilde{\nu} = \mu_1, \dots \mu_a$, and $\beta = \mu_{a+1}$, the previous equation reads as $C^{\mu_0,\dots,\mu_{a+1}}_i = (-1)^{a+1}C^{\mu_{a+1}\mu_0,\dots,\mu_a}_i$, which means (using that the multi-index $\widetilde{\mu}$ is skew-symmetric) that $C^{\mu\widetilde{\mu}}$ is skew-symmetric in all its indices and thus we may write

$$\begin{aligned} C^{\mu\widetilde{\mu}}_i\mathrm{d}A^i_\mu \wedge \mathrm{d}^{a-1}x_{\widetilde{\mu}} &= C^{\widetilde{\nu}}_i\delta^{\mu\widetilde{\mu}}_{\widetilde{\nu}}\mathrm{d}A^i_\mu \wedge \mathrm{d}^{a-1}x_{\widetilde{\mu}} \\ &= C^{\nu_1,\dots,\nu_a}_i\sum_{j=1}^a(-1)^{j-1}\mathrm{d}A^i_{\nu_j} \wedge \mathrm{d}^{a-1}x_{\nu_1,\dots,\hat{\nu_j},\dots,\nu_a}. \end{aligned}$$

Now, contracting with

$$U = \frac{\partial}{\partial\widetilde{p}^{\alpha\beta}_i} \wedge \frac{\partial^n}{\partial^{n-1}x_\gamma},$$

where $\gamma \neq \alpha$ and $\gamma \neq \beta$ we get the condition $D^{i\widetilde{\mu}}_{\alpha\beta}\delta^{\widetilde{\nu}\gamma}_{\widetilde{\mu}} = 0$. This means that $D^{i\widetilde{\mu}}_{\alpha,\beta} = 0$ whenever $\widetilde{\mu}$ contains an index different from α and β. This last observation clearly implies that all coefficients are zero when $|\widetilde{\mu}| \geq 3$ or, equivalently, when $a < n-1$. This implies the first item. Now, for the second, we only need to worry about the case $n = n-1$, and a quick computation shows that the only possibility is to have $C^{i\mu\nu}_{\alpha\beta} = \delta^{\mu\nu}_{\alpha\beta}C$, for certain constant C, so that $C^{i\mu\nu}_{\alpha\beta}\mathrm{d}\widetilde{p}^{\alpha\beta}_i \wedge \mathrm{d}^{n-2}x_{\mu\nu}$ is necessarily a multiple of $\mathrm{d}\widetilde{p}^{\alpha\beta}_i \wedge \mathrm{d}^{n-2}x_{\alpha\beta}$, finishing the proof. □

Remark 5.2.1. Again, for completion, we would like to add the associated multivectors from Lemma 4.2.3. It is enough to choose:

(i) For the form $\sum_{j=1}^a(-1)^{j+1}\mathrm{d}A^i_{\nu_j} \wedge \mathrm{d}^{a-1}x_{\nu_1,\dots,\widehat{\nu_j},\dots,\nu_a}$ we need to take

$$U = \frac{1}{n+2-a}\sum_{i<j}(-1)^{i+j}\frac{\partial}{\partial\widetilde{p}^{\nu_i\nu_j}_i} \wedge \frac{\partial^{n-a}}{\partial^{n-a}x^{\nu_1,\dots,\widehat{\nu_i},\dots,\widehat{\nu_j},\dots,\nu_{n+2-a}}},$$

as an easy computation shows.

(ii) For the form $\mathrm{d}\widetilde{p}^{\alpha\beta}_i \wedge \mathrm{d}^{n-2}x_{\alpha\beta}$, we need to take

$$U = \frac{\partial}{\partial A^i_\alpha} \wedge \frac{\partial}{\partial x^\alpha},$$

which, after another quick computation is seen to satisfy the conditions corresponding requirements.

Finally, to end our discussion, we compute the graded Poisson brackets of some local generators of the algebra of special Hamiltonian forms. The nonzero brackets are given by

$$\begin{aligned}
&\left\{\sum_{j=1}^{n+2-a}(-1)^{j+1}A^i_{\nu_j}\mathrm{d}^{a-1}x_{\nu_1,\ldots,\widehat{\nu_j},\ldots,\nu_{n+2-a}}, p_j^{\alpha\beta}\mathrm{d}^{n-2}x_{\alpha\beta}\right\}\\
&\quad = -\sum_{j=1}^{n+2-a}(-1)^{j+1}\mathrm{d}^{a-2}x_{\nu_1,\ldots,\widehat{\nu_j},\ldots,\nu_{n+2-a},\nu_j}\\
&\quad = (-1)^a(n+2-a)\delta^i_j\mathrm{d}^{a-2}x_{\nu_1,\ldots,\nu_{n+2-a}},\\
&\left\{\sum_{j=1}^{n+2-a}(-1)^{j+1}A^i_{\nu_j}\mathrm{d}^{a-1}x_{\nu_1,\ldots,\widehat{\nu_j},\ldots,\nu_{n+2-a}}, \widetilde{p}_i^{\mu\nu}\mathrm{d}^{n-1}x_{\nu}\right\}\\
&\quad = \sum_{j=1}^{n+2-a}(-1)^{j+1}\delta^{\nu}_{\nu_j}\mathrm{d}^{a-1}x_{\nu_1,\ldots,\widehat{\nu_j},\ldots,\nu_{n+2-a}}.
\end{aligned}$$

6. Conclusions and Further Work

In this paper, we have analyzed the extensions of graded Poisson brackets in classical field theories in order to give a description of the evolution of forms. This study leads to the definition of special Hamiltonian forms, a particular subfamily of Hamiltonian forms that is proved to be a subalgebra. This subalgebra is the subalgebra of forms with defined evolution. To showcase the theory, we compute it (computing the possible exterior derivatives) and the brackets of some generators of this algebra in the case of regular Lagrangians and in the case of Yang–Mills theories, although the theory works with any almost regular Lagrangian.

There are several aspects that remain unsolved, and that we propose as interesting future questions which we were not able to answer at this moment:

(i) As we showed, the bracket of special Hamiltonian forms and Hamiltonians is independent of the extension chosen, so they seem a natural algebra to restrict the Poisson brackets to. We propose to investigate the properties of this algebra in general because, as we showed, this algebra can be associated to any almost regular classical field theory.

(ii) Determining the evolution of arbitrary Hamiltonian forms suggests to study the relationship between these brackets and the constraint algorithm (see [19] for the general geometric theory and [30] for several examples). We believe that the theory behind the constraint algorithm would benefit from an analysis employing the extension of the brackets presented in our study.

(iii) We would also find it interesting to investigate the relationship of these constructions with the Poisson brackets of the canonical formalism (see [38, 39]).

(iv) As we mentioned in Sec. 2, the tools developed may be of interest to the theory of reduction and momentum maps, which is still in full development in the case of classical fields (see [4,5] for recent advances in the case of reduction and [10, 24] for advances in momentum maps).

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